Lecture notes for “Category Theory”

The following text is a reworking by Benno van den Berg of Jaap van Oosten's lecture notes "Basic Category Theory and Topos Theory".

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1 Categories

1.1 Categories

Definition 1.1 A *category* $\mathcal{C}$ is given by a collection $\mathcal{C}_0$ of *objects* and a collection $\mathcal{C}_1$ of *arrows* which have the following structure.

- Each arrow has a *domain* and a *codomain* which are objects; one writes $f{:}\,X \to Y$ or $X \xrightarrow{f} Y$ if X is the domain of the arrow f, and Y its codomain. One also writes $X = \mathrm{dom}(f)$ and $Y = \mathrm{cod}(f)$;
- Given two arrows f and g such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, the *composition* of f and g, written gf, is defined and has domain $\mathrm{dom}(f)$ and codomain $\mathrm{cod}(g)$:
$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto (X \xrightarrow{gf} Z)$$
- Composition is *associative*, that is: given $f{:}\,X \to Y$, $g{:}\,Y \to Z$ and $h{:}\,Z \to W$, $h(gf) = (hg)f$;
- For every object X there is an *identity* arrow $1_X{:}\,X \to X$, satisfying $1_X g = g$ for every $g{:}\,Y \to X$ and $f1_X = f$ for every $f{:}\,X \to Y$.

Instead of "arrow" we also use the terms "morphism" or "map".

Examples

a) $\mathbf{1}$ is the category with one object $*$ and one arrow, 1_*;

b) $\mathbf{0}$ is the empty category. It has no objects and no arrows.

c) A *preorder* is a set X together with a binary relation $\leq$ which is reflexive (i.e. $x \leq x$ for all $x \in X$) and transitive (i.e. $x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in X$). This can be viewed as a category, with set of objects X and for every pair of objects (x, y) such that $x \leq y$, exactly one arrow: $x \to y$. Indeed, we can think of preorders as categories with a set of objects and which are such that for any two objects X, Y of $\mathcal{C}$ there is at most one arrow: $X \to Y$.

d) A *monoid* is a set X together with a binary operation, written like multiplication: xy for $x, y \in X$, which is associative and has a unit element $e \in X$, satisfying $ex = xe = x$ for all $x \in X$. Indeed, a monoid can be seen as a special kind of category in which there is precisely one object.

e) Every directed graph can be made into a category as follows: the objects are the vertices of the graph and the arrows are paths in the graph. We call this the category *freely generated* by the graph D.

f) Set is the category which has the class of all sets as objects, and functions between sets as arrows.

Most basic categories have as objects certain mathematical structures, and the structure-preserving functions as morphisms. Examples:

f) Top is the category of topological spaces and continuous functions.

g) Grp is the category of groups and group homomorphisms.

h) Rng is the category of rings and ring homomorphisms.

i) Grph is the category of graphs and graph homomorphisms.

j) Pos is the category of partially ordered sets and monotone functions.

1.2 Some special arrows

We introduce three classes of maps which can be thought of as generalisations of the notion of a *bijection*, an *injection* and a *surjection*, respectively.

Definition 1.2 A morphism $f\colon A \to B$ is an *isomorphism* (often abbreviated as *iso*) if there is $g\colon B \to A$ such that $fg = 1_B$ and $gf = 1_A$. We call g the *inverse* of f (and vice versa, of course); it is unique if it exists. We also write $g = f^{-1}$.

If an isomorphism $f\colon A \to B$ exists, we call the objects A and B *isomorphic*. We denote this as $A \cong B$.

Definition 1.3 We call an arrow $f\colon A \to B$ *monomorphism* (often abbreviated as *mono*) in a category $\mathcal{C}$, if for any object C and for any pair of arrows $g, h\colon C \to A$, $fg = fh$ implies $g = h$.

Definition 1.4 We call an arrow $f\colon A \to B$ an *epimorphism* (often abbreviated as *epi*, epimorphic) if for any object C and any pair of maps $g, h\colon B \to C$, $gf = hf$ implies $g = h$.

Proposition 1.5 *In the category Sets, a map $f\colon Y \to X$ is ...*

1. *a bijection if and only if it is an iso.*
2. *an injection if and only if it is a mono.*
3. *a surjection if and only if it is an epi.*

Proof. We expect the first statement to be obvious, so we concentrate on the other two. It is not hard to see that injections are mono and surjections epi, so we concentrate on the other direction.

So suppose $f: Y \to X$ is a mono in Sets and $y_0, y_1 \in Y$ are elements in Y such that $f(y_0) = f(y_1)$. Then let $1 = \{*\}$ be a set with a single element $*$ and let $g, h: 1 \to Y$ be the maps with $g(*) = y_0$ and $h(*) = y_1$. Then $fg = fh$ and $g = h$, because f is mono. Therefore $y_0 = y_1$ and we conclude that f is injective.

Finally, suppose $f: Y \to X$ is epi in Sets. Consider the set $\{0, 1\}$, the maps $g: X \to \{0, 1\}$ which is constant 1 and the map $h: X \to \{0, 1\}$ which sends every $x \in X$ which lies in the image of f to 1 and all the other elements to 0. Then $gf = hf$ and $g = h$, because g is epi. Therefore h is the constant 1 function and f is surjective. ∎

You can try to characterise the monos and epis in some of your favourite categories. It is often a good guess that the monos are the injective morphisms (this is true in Grp, Grph, Rng, Preord, Pos, for instance), while the epis are the surjective morphisms. But especially to the second statement many counterexamples exist.

Example. In Mon, the embedding $\mathbb{N} \to \mathbb{Z}$ is an epimorphism.

For, suppose $\mathbb{Z} \underset{g}{\overset{f}{\rightrightarrows}} (M, e, \star)$ two monoid homomorphisms which agree on the nonnegative integers. Then

$$f(-1) = f(-1) \star g(1) \star g(-1) = f(-1) \star f(1) \star g(-1) = g(-1)$$

so f and g agree on the whole of $\mathbb{Z}$. Since $\mathbb{N} \to \mathbb{Z}$ is a mono, this shows that in Mon there are maps which are both mono and epi, without being iso (that is, Mon is not *balanced*).

1.3 Exercises

Exercise 1 Show that 1_X is the *unique* arrow with domain X and codomain X with the property that $f1_X = f$ for every $f: X \to Y$ and $1_X g = g$ for every $g: Y \to X$.

Exercise 2 Let Rel be the category whose objects are sets and whose morphisms $A \to B$ are relations $R \subseteq A \times B$. Here composition of relations is defined as follows: if $R \subseteq A \times B$ and $S \subseteq B \times C$, then

$$S \circ R = \{(a, c) : (\exists b \in B)\, (a, b) \in R \text{ and } (b, c) \in S\}.$$

Show that Rel is a category.

Exercise 3 Show that if two of f, g and fg are iso, then so is the third.

Exercise 4 Show that "being isomorphic" is an equivalence relation on the collection of objects of a category.

Exercise 5 Show that if $g: Y \to X$ and $f: Z \to Y$ are monos, then so is $g \circ f$. Show that if $g \circ f$ is mono, then so is f. What are the corresponding statements for epis?

Exercise 6 A map $f: A \to B$ is called a *split epi* if there is $g: B \to A$ such that $fg = 1_B$ (other names: in this case g is called a *section* of f, and f a *retraction* of g).

(a) Show that isos are split epis and split epis are epi.

(b) Show that a map is an iso if and only if it is both a mono and a split epi.

2 Functors and constructions on categories

2.1 Functors

An important maxim in category theory is that every mathematical structure comes with an appropriate notion of a structure-preserving map. When applying this idea to the notion of category itself, we obtain the concept of a *functor*.

Definition 2.1 Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F: \mathcal{C} \to \mathcal{D}$ consists of operations $F_0: \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1: \mathcal{C}_1 \to \mathcal{D}_1$, such that for each $f: X \to Y$, $F_1(f): F_0(X) \to F_0(Y)$ and:

- for $X \xrightarrow{f} Y \xrightarrow{g} Z$, $F_1(gf) = F_1(g)F_1(f)$;
- $F_1(1_X) = 1_{F_0(X)}$ for each $X \in \mathcal{C}_0$.

But usually we write just F instead of F_0, F_1.

Examples.

a) There is a functor U: Top $\to$ Set which assigns to any topological space X its underlying set. We call this functor "forgetful": it "forgets" the mathematical structure. Similarly, there are forgetful functors Grp $\to$ Set, Grph $\to$ Set, Rng $\to$ Set, Pos $\to$ Set etcetera;

b) For every category $\mathcal{C}$ there is a unique functor $\mathcal{C} \to \mathbf{1}$ and a unique one $\mathbf{0} \to \mathcal{C}$;

c) Given a partially ordered set $(X, \leq)$ we make a topological space by defining $U \subseteq X$ to be open iff for all $x, y \in X$, $x \leq y$ and $x \in U$ imply $y \in U$ (U is "upwards closed", or an "upper set"). This is a topology, called the *Alexandroff topology* w.r.t. the order $\leq$.

   If $(X, \leq)$ and $(Y, \leq)$ are two partially ordered sets, a function $f: X \to Y$ is monotone for the orderings if and only if f is continuous for the Alexandroff topologies. This gives an important functor: Pos $\to$ Top.

d) Given a set A, consider the set $\tilde{A}$ of strings $a_1 \ldots a_n$ on the alphabet $A \cup A^{-1}$ (A^{-1} consists of elements a^{-1} for each element a of A; the sets A and A^{-1} are disjoint and in 1-1 correspondence with each other), such that for no $x \in A$, xx^{-1} or $x^{-1}x$ is a substring of $a_1 \ldots a_n$. Given two such strings $\vec{a} = a_1 \ldots a_n$, $\vec{b} = b_1 \ldots b_m$, let $\vec{a} \star \vec{b}$ the string formed by first taking $a_1 \ldots a_n b_1 \ldots b_m$ and then removing from this string, successively, substrings of form xx^{-1} or $x^{-1}x$, until one has an element of $\tilde{A}$.

   This defines a group structure on $\tilde{A}$. $\tilde{A}$ is called the *free group* on the set A. Please check this and prove that the assignment $A \mapsto \tilde{A}$ is part of a functor: Set $\to$ Grp. This functor is called the *free functor*.

Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ one can define the composition $GF : \mathcal{C} \to \mathcal{E}$. This composition is of course associative and since we have, for any category $\mathcal{C}$, the *identity functor* $\mathcal{C} \to \mathcal{C}$, we have a category Cat which has categories as objects and functors as morphisms.

Examples.

a) If Mon is the category of monoids and Preorder the category of preorders, then we have functors Mon $\to$ Cat and Preorder $\to$ Cat.

b) The operation which assigns to each directed graph the category freely generated by it is part of a functor from the category Dgrph of directed graphs to Cat.

Definition 2.2 We say a functor F *preserves* a property P if whenever an object or arrow (or...) has property P, its F-image does as well.

Proposition 2.3 *Every functor preserves isomorphisms.*

Proof. If $g : X \to Y$ is the inverse of a map $f : Y \to X$, then Fg is the inverse of Ff, because

$$Fg \circ Ff = F(g \circ f) = F(1_Y) = 1_{FY}$$

and, in a similar way, $Ff \circ Fg = 1_{FX}$. ∎

Now a functor does not in general preserve monos or epis: the example of Mon shows that the forgetful functor Mon $\to$ Set does not preserve epis.

Definition 2.4 A functor F *reflects* a property P if whenever the F-image of something (object, arrow,...) has P, then that something has.

Definition 2.5 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is *full* if for every each pair of objects A, B of $\mathcal{C}$, the function

$$F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$$

is a surjection. If it is injective for each pair of objects A, B, we call F *faithful*. If it is bijective for each pair of objects A, B, we call F *fully faithful*.

Proposition 2.6 *(a) A faithful functor reflects epis and monos.*

(b) A fully faithful functor reflects isos.

Proof. Exercise. ∎

2.2 New categories from old

If we think of categories as a generalisation of monoids and preorders, then it makes sense that constructions which make sense for monoids and preorders could be generalised to categories. We will some examples of this principle here.

2.2.1 Product category

Given two categories $\mathcal{C}$ and $\mathcal{D}$ we can define the *product category* $\mathcal{C} \times \mathcal{D}$ which has as objects pairs $(C, D) \in \mathcal{C}_0 \times \mathcal{D}_0$, and as arrows:$(C, D) \to (C', D')$ pairs (f, g) with $f: C \to C'$ in $\mathcal{C}$, and $g: D \to D'$ in $\mathcal{D}$. The projections determine functors $\pi_0: \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_1: \mathcal{C} \times \mathcal{D} \to \mathcal{D}$.

In fact, the product of two categories defines a functor Cat $\times$ Cat $\to$ Cat. (Please check!)

2.2.2 Slice categories

Let $\mathcal{C}$ be a category and C an object of $\mathcal{C}$. The *slice category* $\mathcal{C}/C$ has as objects all arrows g which have codomain C. An arrow from $g: D \to C$ to $h: E \to C$ in $\mathcal{C}/C$ is an arrow $k: D \to E$ in $\mathcal{C}$ such that $hk = g$. Draw like:

$$\begin{tikzcd} D \arrow[rr, "k"] \arrow[dr, "g"'] & & E \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

We say that *this diagram commutes* if we mean that $hk = g$.

2.2.3 Opposite category

If we are given a category $\mathcal{C}$ we can form a new category $\mathcal{C}^{\mathrm{op}}$ which has the same objects and arrows as $\mathcal{C}$, but with reversed direction; so if $f: X \to Y$ in $\mathcal{C}$ then $f: Y \to X$ in $\mathcal{C}^{\mathrm{op}}$. To make it notationally clear, write $\bar{f}$ for the arrow $Y \to X$ corresponding to $f: X \to Y$ in C. Composition in $\mathcal{C}^{\mathrm{op}}$ is defined by:

$$\bar{f}\bar{g} = \overline{gf}$$

The *duality principle*, a very important, albeit trivial, principle in category theory, says that any valid statement about categories, involving the properties $P_1, \ldots, P_n$ implies the "dualized" statement (where direction of arrows is reversed) with the P_i replaced by P_i^{op}. We will examples of this principle throughout the course.

Any functor $F: \mathcal{C} \to \mathcal{D}$ gives a functor $F^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. In fact, we have a functor $(-)^{\mathrm{op}}: \mathrm{Cat} \to \mathrm{Cat}$.

Remark 2.7 The definition of epi is "dual" to the definition of mono. That is, f is epi in the category $\mathcal{C}$ if and only if f is mono in $\mathcal{C}^{\mathrm{op}}$, and vice versa. For that reason the following is an example of the duality principle:

Example. If gf is mono, then f is mono. From this, "by duality", if fg is epi, then f is epi. (Please prove this statement!)

2.3 Terminal object

As another example of the duality principle let us discuss *terminal* and *initial objects*.

Definition 2.8 An object X is called *terminal* in a category $\mathcal{C}$ if for any object Y there is exactly one morphism $Y \to X$ in the category $\mathcal{C}$. Dually, X is *initial* if for all Y there is exactly one $X \to Y$.

Theorem 2.9 *If X and X' are two terminal objects, they are isomorphic. In fact, there exists a unique isomorphism between them.*

Proof. Suppose X and X' are terminal. Then because X is terminal, there is a unique map $f\colon X' \to X$, and because X' is terminal, there is a unique map $g\colon X \to X'$. But because X is terminal, we also have that any two maps $X \to X$ are identical, so we also have $fg = 1_X$. Similarly, $gf = 1_{X'}$. In other words, f and g are isomorphisms, and X and X' are isomorphic. ∎

Remark 2.10 Of course, by duality, the same statement is true for initial objects.

2.4 Exercises

Exercise 7 Let $I\colon \mathrm{Preorder} \to \mathrm{Cat}$ be the inclusion functor. Show that there is a functor $J\colon \mathrm{Cat} \to \mathrm{Preorder}$ such that $JI = 1$.

Exercise 8 Given a topological space X, you can define a preorder $\leq_s$ on X (the "specialization ordering") as follows: say $x \leq_s y$ if for all open sets U, if $x \in U$ then $y \in U$. $\leq_s$ is a partial order iff X is a T_0-space.

For many spaces, $\leq_s$ is trivial (in particular when X is T_1) but in case X is for example the Alexandroff topology on a poset $(X, \leq)$, then $x \leq_s y$ iff $x \leq y$.

If $f\colon X \to Y$ is a continuous map of topological spaces then f is monotone w.r.t. the specialization orderings $\leq_s$. Show that this defines a functor $\mathrm{Top} \to \mathrm{Preord}$, where Preord is the category of preorders and monotone functions.

Exercise 9 Let X be the category defined as follows: objects are pairs (I, x) where I is an open interval in $\mathbb{R}$ and $x \in I$. Morphisms $(I, x) \to (J, y)$ are differentiable functions $f\colon I \to J$ such that $f(x) = y$.

Let Y be the (multiplicative) monoid $\mathbb{R}$, considered as a category. Show that the operation which sends an arrow $f\colon (I, x) \to (J, y)$ to $f'(x)$, determines a functor $X \to Y$. On which basic fact of elementary Calculus does this rely?

Exercise 10 ("Abelianization") Let Abgp be the category of abelian groups and homomorphisms. For every group G the subgroup $[G, G]$ generated by all elements of form $aba^{-1}b^{-1}$ is a normal subgroup. $G/[G, G]$ is abelian, and for every group homomorphism $\phi\colon G \to H$ with H abelian, there is a unique homomorphism $\bar{\phi}\colon G/[G, G] \to H$ such that the diagram

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "\phi"] & \\ G/[G,G] \arrow[rr, "\bar{\phi}"'] & & H \end{tikzcd}$$

commutes. Show that this gives a functor: Grp $\to$ Abgp.

Exercise 11 Convince yourself that the assignment $C \mapsto \mathcal{C}/C$ gives rise to a functor $\mathcal{C} \to$ Cat.

Exercise 12 (This example will become important later.) Let $\mathcal{C}$ be a category such that for every pair (X, Y) of objects the collection $\mathcal{C}(X, Y)$ of arrows from X to Y is a set (such $\mathcal{C}$ is called *locally small*). Show that if $\mathcal{C}$ is locally small, then there is a functor (the "Hom functor") $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ Set, assigning to the pair (A, B) of objects of $\mathcal{C}$, the set $\mathcal{C}(A, B)$.

Exercise 13 Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is surjective on objects, that is, for each object D in $\mathcal{D}$ there is an object C in $\mathcal{C}$ such that $FC = D$. Consider the following three statements:

(i) Each morphism g in $\mathcal{D}$ can be written as $Ff_1 \circ Ff_2 \circ \ldots \circ Ff_n$ for some morphisms $f_1, \ldots, f_n$ in $\mathcal{C}$.

(ii) Each morphism g in $\mathcal{D}$ can be written Ff for some morphism f in $\mathcal{C}$.

(iii) F is full.

Then (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Show that none of these implications can be reversed: that is, show that there are examples of functors which are surjective on objects which have property (i), but not (ii), or property (ii), but not (iii).

Exercise 14 Show that a fully faithful functor reflects terminal objects.

3 Natural transformations and equivalences

3.1 Natural transformations

Fix two categories $\mathcal{C}$ and $\mathcal{D}$. A crucial aspect of category theory is that we can give the collection of functors from $\mathcal{C}$ to $\mathcal{D}$ the structure of a category as well. The morphisms in this category are the *natural transformations*.

Definition 3.1 A *natural transformation* between two functors $F, G: \mathcal{C} \to \mathcal{D}$ consists of a family of morphisms $(\mu_C: FC \to GC)_{C \in \mathcal{C}_0}$ indexed by the collection of objects of $\mathcal{C}$, satisfying the following *naturality condition*: for every morphism $f: C \to C'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} FC \arrow[r, "\mu_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\mu_{C'}"'] & GC' \end{tikzcd}$$

commutes in $\mathcal{D}$ (this means $\mu_{C'} \circ Ff = Gf \circ \mu_C$; the diagram above is called the *naturality square*). We write $\mu = (\mu_C)_{C \in \mathcal{C}_0}: F \Rightarrow G$ and we call μ_C the *component at* C of the natural transformation μ.

It is not hard to see that if $\mu: F \Rightarrow G$ and $\nu: G \Rightarrow H$, then we also have a natural transformation $\nu\mu = (\nu_C \mu_C)_C: F \Rightarrow H$, and that this composition operation has identities and is associative. Therefore there exists a *functor category* $[\mathcal{C}, \mathcal{D}]$ of functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between them.

A natural transformation is called a *natural isomorphism* if it is an isomorphism in this functor category. This is equivalent to a simpler property:

Proposition 3.2 *A natural transformation $\mu: F \Rightarrow G$ in $[\mathcal{C}, \mathcal{D}]$ is a natural isomorphism if and only if each component μ_C is an isomorphism.*

Proof. Note that the identity natural transformation is the identity at each component. Hence if $\mu: F \Rightarrow G$ has an inverse $\nu: G \Rightarrow F$, then we have have that each ν_C is the inverse of μ_C. Conversely, if each μ_C is an iso, then the inverse (if it exist) must be defined by $\nu_C = \mu_C^{-1}$. It remains to check that ν, defined in this way, is actually a natural transformation, so satisfies the naturality condition. But note that for each $f: C \to C'$ in $\mathcal{C}$ we have a diagram

$$\begin{tikzcd} FC \arrow[r, "\mu_C"] \arrow[d, "Ff"] & GC \arrow[r, "\nu_C"] \arrow[d, "Gf"] & FC \arrow[d, "Ff"] \\ FC' \arrow[r, "\mu_{C'}"] & GC' \arrow[r, "\nu_{C'}"] & FC' \end{tikzcd}$$

in which the outer rectangle and left hand square commute. Since μ_C is iso and hence epi, the right hand square also commutes. ∎

Proposition 3.3 *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories. Then functor composition is the object part of a functor*

$$\circ: [\mathcal{D}, \mathcal{E}] \times [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{E}].$$

Proof. First of all, we have to define the action of $\circ$ on morphisms, that is, we have to show how, given functors $F, G: \mathcal{C} \to \mathcal{D}$ and $H, K: \mathcal{D} \to \mathcal{E}$ and natural transformations $\mu: F \Rightarrow G$ and $\mu: H \Rightarrow K$, one can define a natural transformation $\nu * \mu: HF \Rightarrow KG$ (this is called *horizontal composition* of natural transformations; the composition of natural transformations in functor categories is often called *vertical composition*). This means that, given an object C in $\mathcal{C}$ we have to define a map $HFC \to KGC$. A priori, there are two candidates, namely the the two composites along the outsides of the square

$$\begin{array}{ccc} HFC & \xrightarrow{H\mu_C} & HGC \\ \downarrow{\scriptstyle \nu_{FC}} & & \downarrow{\scriptstyle \nu_{GC}} \\ KFC & \xrightarrow{K\mu_C} & KGC. \end{array}$$

Fortunately, these composites are equal, because this square is the naturality square for the natural transformation ν at the map $\mu_C: FC \to GC$. Therefore we may define

$$(\nu * \mu)_C := \nu_{GC} \circ H\mu_C = K\mu_C \circ \nu_{FC}.$$

Some lengthy verifications are now in order, something we gladly leave to the reader. (Exercise!) ∎

3.2 Examples of natural transformations

a) Let M and N be two monoids, regarded as categories with one object as in chapter 1. A functor $F: M \to N$ is then just the same as a homomorphism of monoids. Given two such, say $F, G: M \to N$, a natural transformation $F \Rightarrow G$ is (given by) an element n of N such that $nF(x) = G(x)n$ for all $x \in M$;

b) Let P and Q two preorders, regarded as categories. A functor $P \to Q$ is a monotone function, and there exists a unique natural transformation between two such, $F \Rightarrow G$, exactly if $F(x) \leq G(x)$ for all $x \in P$.

c) Let $U: \mathrm{Grp} \to \mathrm{Set}$ denote the forgetful functor, and $F: \mathrm{Set} \to \mathrm{Grp}$ the free functor (see chapter 1). There are natural transformations $\varepsilon: FU \Rightarrow 1_{\mathrm{Grp}}$ and $\eta: 1_{\mathrm{Set}} \Rightarrow UF$.

Given a group G, ε_G takes the string $\sigma = g_1 \ldots g_n$ to the product $g_1 \cdots g_n$ (here, the "formal inverses" g_i^{-1} are interpreted as the real inverses in G!).

Given a set A, $\eta_A(a)$ is the singleton string a.

d) Let $i\colon \mathrm{Abgp} \to \mathrm{Grp}$ be the inclusion functor and $r\colon \mathrm{Grp} \to \mathrm{Abgp}$ the abelianization functor defined in example m) in chapter 1. There is $\varepsilon\colon ri \Rightarrow 1_{\mathrm{Abgp}}$ and $\eta\colon 1_{\mathrm{Grp}} \Rightarrow ir$.

The components η_G of η are the quotient maps $G \to G/[G,G]$; the components of ε are isomorphisms.

e) Every class of arrows of a category $\mathcal{C}$ can be viewed as a natural transformation. Suppose $S \subseteq \mathcal{C}_1$. Let $F(S)$ be the discrete category on S as in the preceding example. There are the two functors $\mathrm{dom}, \mathrm{cod}\colon F(S) \to \mathcal{C}$, giving the domain and the codomain, respectively. For every $f \in S$ we have $f\colon \mathrm{dom}(f) \to \mathrm{cod}(f)$, and the family $(f|f \in S)$ defines a natural transformation: $\mathrm{dom} \Rightarrow \mathrm{cod}$.

f) Let A and B be sets. There are functors $(-) \times A\colon \mathrm{Set} \to \mathrm{Set}$ and $(-) \times B\colon \mathrm{Set} \to \mathrm{Set}$. Any function $f\colon A \to B$ gives a natural transformation $(-) \times A \Rightarrow (-) \times B$.

g) Given categories $\mathcal{C}$, $\mathcal{D}$ and an object D of $\mathcal{D}$, there is the constant functor $\Delta_D\colon \mathcal{C} \to \mathcal{D}$ which assigns D to every object of $\mathcal{C}$ and 1_D to every arrow of $\mathcal{C}$.

Every arrow $f\colon D \to D'$ gives a natural transformation $\Delta_f\colon \Delta_D \Rightarrow \Delta_{D'}$ defined by $(\Delta_f)_C = f$ for each $C \in \mathcal{C}_0$.

h) Let $\mathcal{P}(X)$ denote the power set of a set X: the set of subsets of X. The powerset operation can be extended to a functor $\mathcal{P}\colon \mathrm{Set} \to \mathrm{Set}$. Given a function $f\colon X \to Y$ define $\mathcal{P}(f)$ by $\mathcal{P}(f)(A) = f[A]$, the image of $A \subseteq X$ under f.

There is a natural transformation $\eta\colon 1_{\mathrm{Set}} \Rightarrow \mathcal{P}$ such that $\eta_X(x) = \{x\} \in \mathcal{P}(X)$ for each set X.

There is also a natural transformation $\mu\colon \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}$. Given $A \in \mathcal{P}\mathcal{P}(X)$, so A is a set of subsets of X, we take its union $\bigcup(A)$ which is a subset of X. Put $\mu_X(A) = \bigcup(A)$.

3.3 Equivalence of categories

As will become clear in the following chapters, equality between objects plays only a minor role in category theory. The most important categorical notions are only defined "up to isomorphism". This is in accordance with mathematical practice and with common sense: just renaming all elements of a group does not really give you another group.

However, once we also consider functor categories, it turns out that there is another relation of "sameness" between categories, weaker than isomorphism of categories, and yet preserving all "good" categorical properties. Isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ requires the existence of functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ such that $FG = 1_{\mathcal{D}}$ and $GF = 1_{\mathcal{C}}$; but bearing in mind that we can't really say meaningful things about equality between objects, we may relax the requirement by just asking that FG is *isomorphic* to $1_{\mathcal{D}}$ in the functor category $[\mathcal{D}, \mathcal{D}]$ (and the same for GF); doing this we arrive at the notion of *equivalence of categories*, which is generally regarded as the proper notion of sameness for categories.

Definition 3.4 A functor $F: \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there is a functor $G: \mathcal{D} \to \mathcal{C}$ such that GF is naturally isomorphic to $1_{\mathcal{C}}$ and FG is naturally isomorphic to $1_{\mathcal{D}}$. The functor G is of course also an equivalence and called a *pseudo-inverse* of F. If such an equivalence F exists, we call the categories $\mathcal{C}$ and $\mathcal{D}$ *equivalent*, something we write as $\mathcal{C} \simeq \mathcal{D}$.

As a simple example of an equivalence of categories, take a preorder P. Let Q be the quotient of P by the equivalence relation which contains the pair (x, y) iff both $x \leq y$ and $y \leq x$ in P. Let $\pi: P \to Q$ be the quotient map. Regarding P and Q as categories, π is a functor, and in fact an equivalence of categories, though not in general an isomorphism.

For recognising equivalences the following result is often useful. It requires a definition.

Definition 3.5 Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is *essentially surjective* if for each object D in $\mathcal{D}$ there is an object C in $\mathcal{C}$ such that $FC \cong D$.

Theorem 3.6 *A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is full, faithful and essentially surjective.*

Proof. If $F: \mathcal{C} \to \mathcal{D}$ has a pseudo-inverse $G: \mathcal{D} \to \mathcal{C}$, we have for each object D in $\mathcal{D}$ that $FGD \cong D$, so F is essentially surjective.

Furthermore, if we have a natural isomorphism $\mu: 1 \Rightarrow GF$, which means that for each pair of objects A, B in $\mathcal{C}$ there is an operation

$$I_{A,B}: D(FA, FB) \to \mathcal{C}(A, B)$$

obtained by sending $g: FA \to FB$ to $\mu_B^{-1} \circ Gg \circ \mu_A$. Naturality of μ gives us that $I_{A,B} \circ F_{A,B} = 1$, so $F_{A,B}$ is injective and equivalences like F and G are faithful.

To show that $F_{A,B} \circ I_{A,B} = 1$, we take a map $g: FA \to FB$ and we have to show $Ff = g$ where f is the unique map filling

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & GFA \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle Gg} \\ B & \xrightarrow{\mu_B} & GFB \end{array}$$

But because μ is a natural isomorphism, we must have that $Gg = GFf$, so we have $g = Ff$, because G, as an equivalence, is faithful.

Conversely, suppose that $F: \mathcal{C} \to \mathcal{D}$ is full, faithful and essentially surjective. Then because F is essentially surjective, there is for each object D in $\mathcal{D}$ there is an object GD in $\mathcal{C}$ together with an isomorphism $\nu_D: D \cong FGD$. Then because F is full and faithful, there is for each $g: D \to D'$ in $\mathcal{D}$ a unique map $Gg: GD \to GD'$ making

$$\begin{array}{ccc} D & \xrightarrow{\nu_D} & FGD \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle FGg} \\ D' & \xrightarrow{\nu_{D'}} & FGD' \end{array}$$

From this it follows that G is a functor and ν a natural isomorphism $1_{\mathcal{D}} \Rightarrow FG$.

It remains to show that there is a natural isomorphism $\mu: 1 \Rightarrow GF$. To construct an isomorphism $\mu_C: C \cong GFC$ it suffices to construct an isomorphism $FC \cong FGFC$, because F is full and faithful (see Exercise 12 below): so we choose μ_C such that $F\mu_C = \nu_{FC}$.

We still have to prove that μ is natural, so let $f: C \to C'$ be any morphism in $\mathcal{C}$ and consider the naturality square:

$$\begin{array}{ccc} C & \xrightarrow{\mu_C} & GFC \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle GFf} \\ C' & \xrightarrow{\mu_{C'}} & GFC'. \end{array}$$

To see that it commutes, it suffices to show that its image under F commutes (F is faithful). But its F-image is

$$\begin{array}{ccc} FC & \xrightarrow{\nu_{FC}} & GFC \\ \Big\downarrow{\scriptstyle Ff} & & \Big\downarrow{\scriptstyle FGFf} \\ FC' & \xrightarrow{\nu_{FC'}} & GFC', \end{array}$$

which commutes because ν is natural. ∎

3.4 Exercises

Exercise 15 There are at least two ways to associate a category to a set X: let $F(X)$ be the category with as objects the elements of X, and as only arrows identities (a category of the form $F(X)$ is called *discrete*; and $G(X)$ the category with the same objects but with exactly one arrow $f_{x,y}\colon x \to y$ for each pair (x, y) of elements of X (We might call $G(X)$ an *indiscrete category*). Check that F and G can be extended to functors: Set $\to$ Cat and describe the natural transformation $\mu\colon F \Rightarrow G$ which has, at each component, the identity function on objects.

Exercise 16 Show that there is an "evaluation functor" $\mathcal{C} \times [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ which on the object level sends an object C in $\mathcal{C}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ to the object FC in $\mathcal{D}$.

Exercise 17 Suppose that F, G, H are functors such that $H = G \circ F$. Show that if two of F, G, H are equivalences, then so is the third. Deduce that "being equivalent categories" is an equivalence relation on the collection of all categories.

Exercise 18 A category in which every arrow is invertible is called a *groupoid*. Show that a category is equivalent to a discrete category if and only if it is a groupoid and a preorder.

Exercise 19 A category is called *skeletal* if any two isomorphic objects are identical. Show that every category is equivalent to a skeletal category.

Exercise 20 Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence. Show that any two pseudo-inverses of F are naturally isomorphic. (This is often expressed as: pseudo-universes are unique up to natural isomorphism.)

4 Limits and colimits

4.1 Limits

In what follows, a functor $F: \mathcal{C} \to \mathcal{D}$ will also be called a *diagram* in $\mathcal{D}$ *of type* $\mathcal{C}$, and $\mathcal{C}$ is the *shape* or *index category* of the diagram. (In this context the category $\mathcal{C}$ is almost always assumed to be small.)

Note that if $D \in \mathcal{D}$ is some object, there is always a diagram of shape $\mathcal{C}$ in $\mathcal{D}$, obtained by sending every object in $\mathcal{C}$ to D and every arrow in $\mathcal{C}$ to the identity on D. We will call this the *constant diagram of shape* $\mathcal{C}$ *at* D and we will denote it by $\Delta_D: \mathcal{C} \to \mathcal{D}$. If $g: D \to D'$ is a morphism, then this induces a natural transformation $\Delta_g: \Delta_D \Rightarrow \Delta_{D'}$ with $(\Delta_g)_C = g$. In fact, Δ can be seen as a functor $\Delta: \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$.

Definition 4.1 Fix a functor $F: \mathcal{C} \to \mathcal{D}$. A *cone* for the diagram F consists of an object D of $\mathcal{D}$ together with a natural transformation $\mu: \Delta_D \Rightarrow F$. In other words, we have a family $(\mu_C: D \to F(C) | C \in \mathcal{C}_0)$, and the naturality requirement in this case means that for every arrow $f: C \to C'$ in $\mathcal{C}$,

$$\begin{tikzcd} & D \arrow[dl, "\mu_C"'] \arrow[dr, "\mu_{C'}"] & \\ F(C) \arrow[rr, "F(f)"'] & & F(C') \end{tikzcd}$$

commutes in $\mathcal{D}$ (this diagram explains, I hope, the name "cone"). Let us denote the cone by (D, μ). D is called the *vertex* of the cone.

A *map of cones* $(D, \mu) \to (D', \mu')$ is a map $g: D \to D'$ such that $\mu' \circ \Delta_g = \mu$. Using that Δ is a functor, one can easily see that there is a category $\mathrm{Cone}(F)$ which has as objects the cones for F and as morphisms maps of cones.

Definition 4.2 A *limiting cone* for F is a terminal object in $\mathrm{Cone}(F)$. Since terminal objects are unique up to unique isomorphism, as we have seen, any two limiting cones are isomorphic in $\mathrm{Cone}(F)$ and in particular, their vertices are isomorphic in $\mathcal{D}$.

Let us see what it means to be a limiting cone, in some simple, but important cases.

4.1.1 Terminal objects, again

A limiting cone for the unique functor $!: \mathbf{0} \to \mathcal{D}$ ($\mathbf{0}$ is the empty category) "is" a terminal object in $\mathcal{D}$. For every object D of $\mathcal{D}$ determines, together with the empty family, a cone for !, and a map of cones is just an arrow in $\mathcal{D}$. So $\mathrm{Cone}(!)$ is isomorphic to $\mathcal{D}$.

4.1.2 Binary products

Let $\mathbf{2}$ be the discrete category with two objects x, y. A functor $\mathbf{2} \to \mathcal{D}$ is just a pair $\langle A, B \rangle$ of objects of $\mathcal{D}$, and a cone for this functor consists of an object C of $\mathcal{D}$ and two maps
$$\begin{tikzcd} C \arrow[r, "\mu_A"] \arrow[dr, "\mu_B"'] & A \\ & B \end{tikzcd}$$
since there are no nontrivial arrows in $\mathbf{2}$.

$(C, (\mu_A, \mu_B))$ is a limiting cone for $\langle A, B \rangle$ iff the following holds: for any object D and arrows $f\colon D \to A$, $g\colon D \to B$, there is a unique arrow $h\colon D \to C$ such that

$$\begin{tikzcd} & D \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ A & C \arrow[l, "\mu_A"] \arrow[r, "\mu_B"'] & B \end{tikzcd}$$

commutes. In other words, there is, for any D, a 1-1 correspondence between maps $D \to C$ and pairs of maps
$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$
This is the property of a *product*; a limiting cone for $\langle A, B \rangle$ is therefore called a product cone, and usually denoted:

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

The arrows π_A and π_B are called *projections*.

4.1.3 Equalizers

Let $\hat{\mathbf{2}}$ denote the category $x \overset{a}{\underset{b}{\rightrightarrows}} y$. A functor $\hat{\mathbf{2}} \to \mathcal{D}$ is the same thing as a parallel pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{D}$; I write $\langle f, g \rangle$ for this functor. A cone for $\langle f, g \rangle$ is:

$$\begin{tikzcd} & D \arrow[dl, "\mu_A"'] \arrow[dr, "\mu_B"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

But $\mu_B = f\mu_A = g\mu_A$ is already defined from μ_A, so giving a cone is the same as giving a map $\mu_A\colon D \to A$ such that $f\mu_A = g\mu_A$. Such a cone is limiting iff for any other map $h: C \to A$ with $fh = gh$, there is a unique $k: C \to D$ such that $h = \mu_A k$.

We call μ_A, if it is limiting, an *equalizer* of the pair f, g, and the diagram $D \xrightarrow{\mu_A} A \underset{g}{\overset{f}{\rightrightarrows}} B$ an equalizer diagram.

In Sets, an equalizer of f, g is isomorphic (as a cone) to the inclusion of $\{a \in A | f(a) = g(a)\}$ into A. In categorical interpretations of logical systems, equalizers are used to interpret *equality between terms*.

4.1.4 Pullbacks

Let J denote the category
$$\begin{tikzcd} & y \arrow[d, "b"] \\ x \arrow[r, "a"'] & z \end{tikzcd}$$
A functor $F: J \to \mathcal{D}$ is specified by giving two arrows in $\mathcal{D}$ with the same codomain, say $f: X \to Z$, $g: Y \to Z$. A limit for such a functor is given by an object W together with projections
$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \\ X & \end{tikzcd}$$
satisfying $fp_X = gp_Y$, and such that, given any other pair of arrows:
$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "s"'] & Y \\ X & \end{tikzcd}$$
with $gr = fs$, there is a unique arrow $V \to W$ such that

$$\begin{tikzcd} V \arrow[dr] \arrow[drr, "r"] \arrow[ddr, "s"'] & & \\ & W \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes.

The diagram
$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
is called a *pullback diagram.* In Set, the pullback cone for f, g is isomorphic to
$$\{(x, y) \in X \times Y | f(x) = g(y)\}$$

with the obvious projections.

4.1.5 Some more terminology

Definition 4.3 We say that a category $\mathcal{C}$ *has limits of shape* $\mathcal{I}$ if a limiting cone exists for each diagram $F: \mathcal{I} \to \mathcal{C}$. If a category $\mathcal{C}$ has limits of shape $\mathcal{I}$ for all small categories $\mathcal{I}$, we say that $\mathcal{C}$ is *complete*. If a category $\mathcal{C}$ has limits of shape $\mathcal{I}$ for all finite categories $\mathcal{I}$ (which means: there are only finitely many objects and arrows in $\mathcal{I}$), then we say that $\mathcal{D}$ is *finitely complete* (or *lex* (*left exact*), or *cartesian*).

So, for instance, we say that a category $\mathcal{C}$ *has binary products* (equalizers, pullbacks) iff every functor $\mathbf{2} \to \mathcal{C}$ ($\hat{\mathbf{2}} \to \mathcal{C}$, $J \to \mathcal{C}$, respectively) has a limiting cone.

Definition 4.4 Let (C, μ) be a limiting cone for a diagram $M: \mathcal{I} \to \mathcal{C}$. We say that this *limit is preserved by* a functor $F: \mathcal{C} \to \mathcal{D}$ if $(FC, F\mu = (F(\mu_I)|I \in \mathcal{I}_0))$ is a limiting cone for FM in $\mathcal{D}$. We say that $F: \mathcal{C} \to \mathcal{D}$ preserves limits of shape $\mathcal{I}$ if it preserves any limiting cone for any diagram $\mathcal{I} \to \mathcal{C}$; and we say that it preserves small (or finite) limits if it preserves limits of any small (or finite) shape $\mathcal{I}$.

So, a functor $F: \mathcal{C} \to \mathcal{D}$ preserves binary products if for every product diagram
$$\begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \\ A \end{tikzcd}$$
its F-image
$$\begin{tikzcd} F(A \times B) \arrow[r, "F(\pi_B)"] \arrow[d, "F(\pi_A)"'] & F(B) \\ F(A) \end{tikzcd}$$
is again a product diagram. Similarly for equalizers and pullbacks.

4.2 Colimits

The dual notion of limit is colimit. Given a functor $F: \mathcal{E} \to \mathcal{C}$ there is clearly a functor $F^{\mathrm{op}}: \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ which does "the same" as F. We say that a *colimiting cocone* for F is a limiting cone for F^{op}.

So: a cocone for $F: \mathcal{E} \to \mathcal{C}$ is a pair (ν, D) where $\nu: F \Rightarrow \Delta_D$ and a colimiting cocone is an initial object in the category $\mathrm{Cocone}(F)$.

Examples

i) a colimiting cocone for $!: \mathbf{0} \to \mathcal{C}$ "is" an initial object of $\mathcal{C}$

ii) a colimiting cocone for $\langle A, B\rangle : \mathbf{2} \to \mathcal{C}$ is a *coproduct* of A and B in $\mathcal{C}$: usually denoted $A + B$ or $A \sqcup B$; there are *coprojections* or *coproduct inclusions*

$$\begin{tikzcd} A \arrow[dr, "\nu_A"] & \\ B \arrow[r, "\nu_B"'] & A \sqcup B \end{tikzcd}$$

with the property that, given any pair of arrows $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ there is a unique map $\begin{bmatrix} f \\ g \end{bmatrix} : A \sqcup B \to C$ such that $f = \begin{bmatrix} f \\ g \end{bmatrix} \nu_A$ and $g = \begin{bmatrix} f \\ g \end{bmatrix} \nu_B$

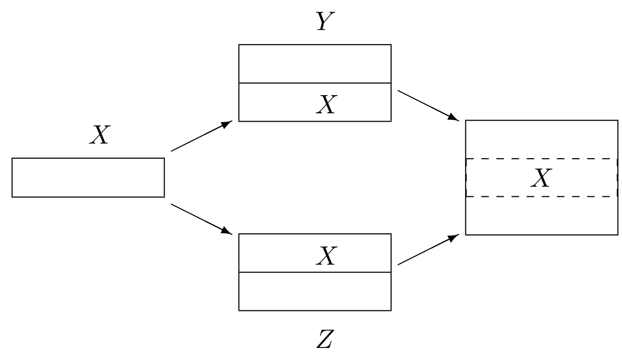
iii) a colimiting cocone for $A \overset{f}{\underset{g}{\rightrightarrows}} B$ (as functor $\hat{\mathbf{2}} \to \mathcal{C}$) is given by a map $B \xrightarrow{c} C$ satisfying $cf = cg$, and such that for any $B \xrightarrow{h} D$ with $hf = hg$ there is a unique $C \xrightarrow{h'} D$ with $h = h'c$. c is called a *coequalizer* for f and g; the diagram $A \rightrightarrows B \longrightarrow C$ a coequalizer diagram.
In Set, the coproduct of X and Y is the disjoint union $(\{0\} \times X) \cup (\{1\} \times Y)$ of X and Y. The coequalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the quotient map $Y \to Y/\sim$ where $\sim$ is the equivalence relation generated by

$$y \sim y' \text{ iff there is } x \in X \text{ with } f(x) = y \text{ and } g(x) = y'$$

iv) The dual notion of pullback is *pushout*. A pushout diagram is a colimiting cocone for a functor $\Gamma \to \mathcal{C}$ where Γ is the category $\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd}$. Such a diagram is a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "a"] \\ Z \arrow[r, "b"'] & P \end{tikzcd}$$

which commutes and such that, given $\begin{tikzcd} & Y \arrow[d, "\alpha"] \\ Z \arrow[r, "\beta"'] & Q \end{tikzcd}$ with $\alpha f = \beta g$, there is a unique $P \xrightarrow{p} Q$ with $\alpha = pa$ and $\beta = pb$. In Set, the pushout of $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ is the coproduct $Y \sqcup Z$ where the two images of X are identified:



> The reader is encouraged to find , in terms of $X \overset{f}{\to} Y$ and $X \overset{g}{\to} Z$, a formal definition of a relation R on $Y \sqcup Z$ such that the pushout of f and g is $Y \sqcup Z/\sim$, $\sim$ being the equivalence relation generated by R.

Also the definitions in Section 3.2.5 can be dualised: for instance, a category $\mathcal{C}$ is *cocomplete* if it has all colimits for diagrams $\mathcal{I} \to \mathcal{C}$ with $\mathcal{I}$ small.

The categories Set, Top, Pos, Mon, Grp, Grph, Rng, Cat ... are all both complete and cocomplete. (This is not supposed to be obvious; it is supposed to be true.)

4.3 Exercises

Exercise 21 Show that a full and faithful functor reflects the property of being a terminal (or initial) object.

Exercise 22 Show that every equalizer is a monomorphism.

Exercise 23 If $E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is an equalizer diagram, show that e is an isomorphism if and only if $f = g$.

Exercise 24 Show that in Set, every monomorphism fits into an equalizer diagram.

Exercise 25 Let

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

a pullback diagram with f mono. Show that a is also mono. Also, if f is iso (an isomorphism), so is a.

Exercise 26 Given two commuting squares:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[r, "c"] \arrow[d, "f"'] & C \arrow[d, "d"'] \\ X \arrow[r, "g"'] & Y \arrow[r, "h"'] & Z \end{tikzcd}$$

a) if both squares are pullback squares, then so is the composite square
$$\begin{tikzcd} A \arrow[r, "cb"] \arrow[d, "a"'] & C \arrow[d, "d"] \\ X \arrow[r, "hg"'] & Z \end{tikzcd}$$

b) If the right hand square and the composite square are pullbacks, then so is the left hand square.

Exercise 27 $f\colon A \to B$ is a monomorphism if and only if

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback diagram.

A monomorphism $f\colon A \to B$ which fits into an equalizer diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

is called a *regular mono*.

Exercise 28 If

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & X \arrow[d, "g"] \\ B \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback and g is regular mono, so is b.

Exercise 29 If f is regular mono and epi, f is iso. Every split mono is regular.

Exercise 30 Give an example of a category in which not every mono is regular.

Exercise 31 In Grp, every mono is regular [This is not so easy].

Exercise 32 Characterize the regular monos in Pos.

Exercise 33 If a category $\mathcal{D}$ has binary products and a terminal object, it has *all finite products*, i.e. limiting cones for every functor into $\mathcal{D}$ from a finite discrete category.

Exercise 34 Suppose $\mathcal{C}$ has binary products and suppose for every ordered pair (A, B) of objects of $\mathcal{C}$ a product cone
$$\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \\ B & \end{tikzcd}$$
has been chosen.

a) Show that there is a functor: $\mathcal{C} \times \mathcal{C} \overset{-\times-}{\to} \mathcal{C}$ (the product functor) which sends each pair (A, B) of objects to $A \times B$ and each pair of arrows $(f: A \to A', g: B \to B')$ to $f \times g = \langle f\pi_A, g\pi_B \rangle$.

b) From a), there are functors:
$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \underset{-\times(-\times-)}{\overset{(-\times-)\times-}{\rightrightarrows}} \mathcal{C}$$
sending (A, B, C) to $\begin{array}{c}(A \times B) \times C \\ A \times (B \times C)\end{array}$ Show that there is a natural transformation $a = (a_{A,B,C} | A, B, C \in \mathcal{C}_0)$ from $(- \times -) \times -$ to $- \times (- \times -)$ such that for any four objects A, B, C, D of $\mathcal{C}$:
$$\begin{tikzcd} ((A \times B) \times C) \times D \arrow[rr, "a_{A\times B,C,D}"] \arrow[d, "a_{A,B,C}\times 1_D"'] & & (A \times B) \times (C \times D) \arrow[d, "a_{A,B,C\times D}"] \\ (A \times (B \times C)) \times D \arrow[dr, "a_{A,B\times C,D}"'] & & A \times (B \times (C \times D)) \\ & A \times ((B \times C) \times D) \arrow[ur, "1_A\times a_{B,C,D}"'] & \end{tikzcd}$$
commutes (This diagram is called "MacLane's pentagon").

Exercise 35 If a category $\mathcal{C}$ has equalizers, it has *all finite equalizers*: for every category $\mathcal{E}$ of the form

$$\begin{array}{ccc} & \xrightarrow{f_1} & \\ X & \vdots & Y \\ & \xrightarrow[f_n]{} & \end{array}$$

every functor $\mathcal{E} \to \mathcal{C}$ has a limiting cone.

Exercise 36 Suppose $F: \mathcal{C} \to \mathcal{D}$ preserves equalizers (and $\mathcal{C}$ has equalizers) and reflects isomorphisms. Then F is faithful.

Exercise 37 Let $\mathcal{C}$ be a category with finite limits. Show that for every object C of $\mathcal{C}$, the slice category $\mathcal{C}/C$ (example j) of 1.1) has binary products: the vertex of a product diagram for two objects $D \to C$, $D' \to C$ is $D'' \to C$ where

$$\begin{array}{ccc} D'' & \longrightarrow & D \\ \downarrow & & \downarrow \\ D' & \longrightarrow & C \end{array}$$

is a pullback square in $\mathcal{C}$.

Exercise 38 Is the terminology "coproduct inclusions" correct? That is, it suggests they are monos. Is this always the case?

Formulate a condition on A and B which implies that ν_A and ν_B are monic.

Exercise 39 Call an arrow f a *stably regular epi* if whenever $\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f} \\ \cdot & \longrightarrow & \cdot \end{array}$ is a pullback diagram, the arrow a is a regular epi. Show: in Pos, $X \xrightarrow{f} Y$ is a stably regular epi if and only if for all y, y' in Y:

$$y \leq y' \Leftrightarrow \exists x \in f^{-1}(y) \exists x' \in f^{-1}(y').x \leq x'$$

Show by an example that not every epi is stably regular in Pos.

Exercise 40 In Grp, every epi is regular.

Exercise 41 Characterize coproducts in Abgrp.

5 Complete categories

Recall from the previous section that a category is called *(co)complete* if it has (co)limits of type $\mathcal{E}$ for all small $\mathcal{E}$. We also claimed that categories Set, Top, Pos, Mon, Grp, Grph, Rng, Cat ... are all both complete and cocomplete. We will not prove that here; instead, we will provide some useful tools for proving statements of that kind. Also, we will establish some properties of complete categories.

Note, by the way, that limits over large (i.e. not small) diagrams need not exist in these examples. For example in Set, there is a limiting cone for the identity functor Set $\to$ Set (its vertex is the empty set), but not for the constant functor $\Delta_A: \mathcal{C} \to$ Set if $\mathcal{C}$ is a large discrete category and A has more than one element.

5.1 Limits by products and equalizers

To prove that a category is complete it suffices to prove that a category has small products and equalizers. (Here small is supposed to include empty, so small products include the empty product, i.e. a terminal object 1.) For instance, in Top, the product of a set $(X_i \,|\, i \in I)$ of topological spaces is the set $\prod_{i\in I} X_i$ with the product topology; the equalizer of two continuous maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the inclusion $X' \subseteq X$ where $X' = \{x \in X \,|\, f(x) = g(x)\}$ with the subspace topology from X. Hence Top is complete.

In Set, every small diagram has a limit; given a functor $F: \mathcal{E} \to$ Set with $\mathcal{E}$ small, there is a limiting cone for F in Set with vertex

$$\{(x_E)_{E\in\mathcal{E}_0} \in \prod_{E\in\mathcal{E}_0} F(E) \,|\, \forall E \xrightarrow{f} E' \in \mathcal{E}_1(F(f)(x_E) = x_{E'})\}$$

So in Set, limits are equationally defined subsets of suitable products. This holds in any category:

Proposition 5.1 *If $\mathcal{C}$ has all small products and equalizers, then $\mathcal{C}$ has all small limits.*

Proof. Given a set I and an I-indexed family of objects $(A_i|i \in I)$ of $\mathcal{C}$, we denote the product by $\prod_{i\in I} A_i$ and projections by $\pi_i: \prod_{i\in I} A_i \to A_i$; an arrow $f: X \to \prod_{i\in I} A_i$ which is determined by the compositions $f_i = \pi_i f: X \to A_i$, is also denoted $(f_i|i \in I)$.

Now given $\mathcal{E} \to \mathcal{C}$ with $\mathcal{E}_0$ and $\mathcal{E}_1$ sets, we construct

$$E \xrightarrow{e} \prod_{i\in\mathcal{E}_0} F(i) \overset{(\pi_{\mathrm{cod}(u)}|u\in\mathcal{E}_1)}{\underset{(F(u)\pi_{\mathrm{dom}(u)}|u\in\mathcal{E}_1)}{\rightrightarrows}} \prod_{u\in\mathcal{E}_1} F(\mathrm{cod}(u))$$

in $\mathcal{C}$ as an equalizer diagram. The family $(\mu_i = \pi_i e: E \to F(i) | i \in \mathcal{E}_0)$ is a natural transformation $\Delta_E \Rightarrow F$ because, given an arrow $u \in \mathcal{E}_1$, say $u: i \to j$, we have that

$$\begin{tikzcd} & E \arrow[dl, "\pi_i e"'] \arrow[dr, "\pi_j e"] & \\ F(i) \arrow[rr, "F(u)"'] & & F(j) \end{tikzcd}$$

commutes since $F(u)\pi_i e = F(u)\pi_{\mathrm{dom}(u)} e = \pi_{\mathrm{cod}(u)} e = \pi_j e$.

So (E, μ) is a cone for F, but every other cone (D, ν) for F gives a map $d: D \to \prod_{i \in \mathcal{E}_0} F(i)$ equalizing the two horizontal arrows; so factors uniquely through E. ∎

Remark 5.2 In the statement of this proposition "small" can be replace by "finite" (or "countable", or ...): if $\mathcal{C}$ has all finite (countable) products and equalizers, then $\mathcal{C}$ has all finite (countable) limits. And, of course, it implies the dual statement: if $\mathcal{C}$ has all small coproducts and coequalizers, then $\mathcal{C}$ has all small colimits.

For proving that a category is finitely complete the following lemma is often useful.

Lemma 5.3 *If a category $\mathcal{D}$ has a terminal object and pullbacks, it has binary products and equalizers.*

Proof. Let 1 be the terminal object in $\mathcal{D}$; given objects X and Y, if $\begin{tikzcd} C \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd}$

is a pullback diagram, then $\begin{tikzcd} C \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \\ Y & \end{tikzcd}$ is a product cone.

Given a product cone $\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \\ B & \end{tikzcd}$ and maps $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & A \\ B & \end{tikzcd}$ we write

$X \overset{\langle f,g \rangle}{\to} A \times B$ for the unique factorization through the product. Write also $\delta: Y \to Y \times Y$ for $\langle 1_Y, 1_Y \rangle$.

Now given $f, g: X \to Y$, if

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d] & X \arrow[d, "\langle f,g \rangle"] \\ Y \arrow[r, "\delta"'] & Y \times Y \end{tikzcd}$$

is a pullback diagram, then $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an equalizer diagram. This proves the lemma. ∎

Corollary 5.4 *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ is finitely complete.*
2. *$\mathcal{C}$ has all finite products and equalizers.*
3. *$\mathcal{C}$ has all pullbacks and a terminal object.*

5.2 Properties of complete categories

Theorem 5.5 *If $\mathcal{D}$ is complete, then so is $[\mathcal{C}, \mathcal{D}]$ for any category $\mathcal{C}$. More generally, if $\mathcal{D}$ has limits of shape $\mathcal{I}$, then so does $[\mathcal{C}, \mathcal{D}]$ for any category $\mathcal{C}$.*

Note that the category $\mathcal{C}$ can be arbitrary (in particular, it need not be complete).

Proof. The slogan is: *(co)limits in functor categories are "computed pointwise"*. That is, let $F: \mathcal{E} \to [\mathcal{C}, \mathcal{D}]$ be a diagram in $[\mathcal{C}, \mathcal{D}]$. Fixing an object $C \in \mathcal{C}_0$ there is a functor $F_C: \mathcal{E} \to \mathcal{D}$, given by $F_C(E) = F(E)(C)$ and for $f: E \to E'$ in $\mathcal{E}$, $F_C(f) = F(f)_C: F(E)(C) \to F(E')(C)$.

Since $\mathcal{D}$ is complete, every F_C has a limiting cone (X_C, μ_C) in $\mathcal{D}$. Now if $C \xrightarrow{g} C'$ is a morphism in $\mathcal{C}$, the collection of arrows

$$\{X_C \overset{(\mu_C)_E}{\to} F(E)(C) \overset{F(E)(g)}{\to} F(E)(C') = F_{C'}(E) \mid E \in \mathcal{E}_0\}$$

is a cone for $F_{C'}$ with vertex X_C, since for any $f: E \to E'$ we have $F(f)_{C'} \circ F(E)(g) \circ (\mu_C)_E = F(E')(g) \circ F(f)_C \circ (\mu_C)_E$ (by naturality of $F(f)$) $= F(E')(g) \circ (\mu_C)_{E'}$ (because (X_C, μ_C) is a cone).

Because $(X_{C'}, \mu_{C'})$ is a limiting cone for $F_{C'}$, there is a unique arrow $X_g: X_C \to X_{C'}$ in $\mathcal{D}$ such that $F(E)(g) \circ (\mu_C)_E = (\mu_{C'})_E \circ X_g$ for all $E \in \mathcal{E}_0$. By the uniqueness of these arrows, we have an object X of $[\mathcal{C}, \mathcal{D}]$, and arrows $\nu_E: X \to F(E)$ for all $E \in \mathcal{E}_0$, and the pair (X, ν) is a limiting cone for F in $[\mathcal{C}, \mathcal{D}]$.

We leave it to the reader to check the remaining details. ∎

Proposition 5.6 *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence and $\mathcal{I}$ be a category. Then F preserves and reflects limits of shape $\mathcal{I}$.*

Proof. Note that if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $M : \mathcal{I} \to \mathcal{C}$ is a diagram, then F induces a functor

$$\widehat{F} : \mathrm{Cone}(M) \to \mathrm{Cone}(FM).$$

Since the morphisms in these categories of cones are certain morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively, and the functor $\widehat{F}$ acts by simply applying F, then functor $\widehat{F}$ will be full and faithful, whenever F is. Since full and faithful functors reflect the terminal object (exercise!), it follows that whenever F is full and faithful, $\widehat{F}$ reflects the terminal object; in other words, full and faithful functors reflect limits.

If F is an equivalence with pseudo-inverse G, then $\widehat{F}$ is also an equivalence with pseudo-inverse $\widehat{G}$. Since equivalences like $\widehat{F}$ preserve the terminal object, it follows that F preserves limits. ∎

Corollary 5.7 *Suppose $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories. If $\mathcal{C}$ is complete, then so is $\mathcal{D}$. More generally, if $\mathcal{C}$ has limits of shape $\mathcal{I}$, then so does $\mathcal{D}$.*

To finish this section a little theorem by Peter Freyd which says that every small, complete category is a complete preorder:

Proposition 5.8 *Suppose $\mathcal{C}$ is small and complete. Then $\mathcal{C}$ is a preorder.*

Proof. If not, there are objects A, B in $\mathcal{C}$ such that there are two distinct maps $f, g : A \to B$. Since $\mathcal{C}_1$ is a set and $\mathcal{C}$ complete, the product $\prod_{h \in \mathcal{C}_1} B$ exists. Arrows $k : A \to \prod_{h \in \mathcal{C}_1} B$ are in 1-1 correspondence with families of arrows $(k_h : A \to B \mid h \in \mathcal{C}_1)$. For every subset $X \subseteq \mathcal{C}_1$ define such a family by:

$$k_h = \begin{cases} f & \text{if } h \in X \\ g & \text{else} \end{cases}$$

This gives an injective function from $2^{\mathcal{C}_1}$ into $\mathcal{C}(A, \prod_{h \in \mathcal{C}_1} B)$ hence into $\mathcal{C}_1$, contradicting Cantor's theorem in set theory. ∎

5.3 Exercises

Exercise 42 Take one of your favourite categories (Top, Pos, Rng, Mon, Grp, Grph, Cat) and show it is both complete and cocomplete.

Exercise 43 Show that if $\mathcal{C}$ is complete, then $F : \mathcal{C} \to \mathcal{D}$ preserves all limits if F preserves products and equalizers. This no longer holds if $\mathcal{C}$ is not complete! That is, F may preserve all products and equalizers which exist in $\mathcal{C}$, yet not preserve all limits which exist in $\mathcal{C}$.

Exercise 44 Suppose a category $\mathcal{C}$ has limits of shape $\mathcal{I}$. Show that the operation which assigns each diagram $\mathcal{I} \to \mathcal{C}$ to its limit in $\mathcal{C}$ in part of a functor

$$[\mathcal{I}, \mathcal{C}] \to \mathcal{C}.$$

Exercise 45 Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. Showing that the following categories are isomorphic:

$$[\mathcal{E}, [\mathcal{C}, \mathcal{D}]] \cong [\mathcal{E} \times \mathcal{C}, \mathcal{D}] \cong [\mathcal{C}, [\mathcal{E}, \mathcal{D}]].$$

Use this and the previous exercise to give a more elegant proof of Theorem 4.5.

Exercise 46 Show that a full and faithful functor reflects the property of being a terminal (or initial) object. Deduce that equivalences preserve the terminal (or initial) object.

6 Cartesian closed categories

Many set-theoretical constructions are completely determined (up to isomorphism, as always) by their categorical properties in Set. We are therefore tempted to generalize them to arbitrary categories, by taking the characteristic categorical property as a definition. Of course, this procedure is not really well-defined and it requires sometimes a real insight to pick the 'right' categorical generalization. For example, the category of sets has very special properties:

- $f: X \to Y$ is mono if and only if $fg = fh$ implies $g = h$ for any two maps $g, h: 1 \to X$, where 1 is a terminal object (we say 1 is *a generator*);
- objects X and Y are isomorphic if there exist monos $f: X \to Y$ and $g: Y \to X$ (the Cantor-Bernstein theorem);
- every mono $X \xrightarrow{f} Y$ is part of a coproduct diagram

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

And if you believe the axiom of choice, there is its categorical version:

- Every epi is split

None of these properties is generally valid, and categorical generalizations based on them are usually of limited value.

In this chapter we focus on a categorical generalization of a set-theoretical concept which has proved to have numerous applications: an exponential as the generalization of "function space".

6.1 Exponentials

Throughout this section we assume we are working in a category $\mathcal{C}$ with chosen finite products: that is, we have chosen a product diagram for every pair of objects of $\mathcal{C}$, as well as a distinguished terminal object 1 in $\mathcal{C}$.

Definition 6.1 Suppose X and Y are objects in $\mathcal{C}$. An *exponential* is an object X^Y in $\mathcal{C}$ together with a morphism $\mathrm{ev}: X^Y \times Y \to X$ (the *evaluation*) such that for each object A and morphism $h: A \times Y \to X$ there is a unique map $H: A \to X^Y$ making

$$\begin{tikzcd} X^Y \times Y \arrow[r, "\mathrm{ev}"] & X \\ A \times Y \arrow[u, "H \times 1_Y"] \arrow[ur, "h"'] & \end{tikzcd}$$

commute. The latter condition can also be formulated as: for each object A the operation

$$\mathrm{Hom}_{\mathcal{C}}(A, X^Y) \to \mathrm{Hom}_{\mathcal{C}}(A \times Y, X) \colon H \mapsto \mathrm{ev} \circ (H \times 1_Y)$$

is a bijection. Morphisms like h and H which correspond to each other under this bijection are called each other's *transposes*.

Definition 6.2 *A category $\mathcal{C}$ is called* cartesian closed *or* a ccc *if it has finite products, and for every pair of objects X and Y in $\mathcal{C}$ an exponential X^Y exists.*

Examples

a) A preorder (or partial order) is cartesian closed if it has a top element 1, binary meets $x \wedge y$ and for any two elements x, y an element $x \to y$ satisfying for each z:
$$z \leq x \to y \text{ iff } z \wedge x \leq y$$

b) Set is cartesian closed, with X^Y being the set of functions $f \colon Y \to X$ and the evaluation map $X^Y \times Y \to X$ being the map which sends a pair (f, y) to $f(y)$.

c) Pos and Preorder are cartesian closed. The exponent Y^X is the set of all monotone maps $X \to Y$, ordered pointwise ($f \leq g$ iff for all $x \in X$, $fx \leq gx$ in Y);

d) **1** is cartesian closed; **0** isn't (why?);

e) A monoid is never cartesian closed unless it is trivial. However, if from the definition of 'cartesian closed' one would delete the requirement that it has a terminal object, an interesting class of 'cartesian closed' monoids exists: the *C-monoids* in the book "Higher Order Categorical Logic" by J. Lambek and Ph. Scott.

f) Top, Grp, Abgp and Mon are not cartesian closed. We will need a bit more theory to see why.

Proposition 6.3 *Cat is cartesian closed with the functor category $[\mathcal{C}, \mathcal{D}]$ acting as the exponential $\mathcal{D}^{\mathcal{C}}$.*

Proof. First of all, we need to define a functor

$$\mathrm{ev}\colon [\mathcal{C}, \mathcal{D}] \times \mathcal{C} \to \mathcal{D}$$

which is defined on objects by sending a pair (F, C) consisting of a functor $F\colon \mathcal{C} \to \mathcal{D}$ and an object C in $\mathcal{C}$ to FC. On arrows it is defined by sending a natural transformation $\sigma\colon F \Rightarrow G$ and a morphism $f\colon C \to C'$ to either composite around the commutative square

$$\begin{tikzcd} FC \arrow[r, "\sigma_C"] \arrow[d, "Ff"] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\sigma_{C'}"] & GC'. \end{tikzcd}$$

We leave the verification that ev is indeed a functor as an exercise.

It remains to check that for each category $\mathcal{E}$ the operation

$$\Phi\colon \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E}, [\mathcal{C}, \mathcal{D}]) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E} \times \mathcal{C}, \mathcal{D})\colon H \mapsto \mathrm{ev}(H \times 1_{\mathcal{C}})$$

defines a bijection between two sets of functors.

To see that Φ is surjective, suppose we are given a functor $F\colon \mathcal{E} \times \mathcal{C} \to \mathcal{D}$. Then define a functor $H\colon \mathcal{E} \to [\mathcal{C}, \mathcal{D}]$ by letting for each object E in $\mathcal{E}$ the functor $H(E)\colon \mathcal{C} \to \mathcal{D}$ be defined by sending C to $F(E, C)$ and $f\colon C \to C'$ to $F(1_E, f)$. In addition, if $g\colon E \to E'$ is a morphism in $\mathcal{E}$, then we can define a natural transformation $H(g)\colon H(E) \Rightarrow H(E')$ by sending each C in $\mathcal{C}$ to the morphism $F(g, 1_C)\colon F(E, C) \to F(E', C)$ in $\mathcal{D}$. This is a natural transformation because for each morphism $f\colon C \to C'$ we have $(g, f) = (1_{E'}, f)(g, 1_C) = (g, 1_{C'})(1_E, f)$ and hence that

$$\begin{tikzcd} F(E, C) \arrow[r, "H(E)(f)"] \arrow[d, "H(g)_C"] & F(E, C') \arrow[d, "H(g)_{C'}"] \\ F(E', C) \arrow[r, "H(E')(f)"'] & F(E', C') \end{tikzcd}$$

commutes. In fact, we can think of $\Psi(F) = H$ as defining an operation

$$\Psi\colon \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E}, [\mathcal{C}, \mathcal{D}]).$$

We leave the verification that Φ and Ψ are each other's inverses to the reader (feel free to use the Bifunctor Lemma, see exercise 4, at this point). ∎

Remark 6.4 In a way, the definition of a natural transformation is precisely what makes the previous proposition work and for that reason we can regard it as an explanation for why natural transformations are defined the way they are. First of all, note that the objects in a category $\mathcal{E}$ are in 1-to-1 correspondence with the arrows $1 \to \mathcal{E}$. Similarly, let $\mathbb{I}$ be the category which looks like this:

$$0 \longrightarrow 1,$$

and let $\varepsilon_i : 1 \to \mathbb{I}$ be the functor which sends the unique object in 1 to the object i in $\mathbb{I}$. Then the arrows from an object A to an object B in a category $\mathcal{E}$ are in 1-to-1 correspondence with the functors $F : \mathbb{I} \to \mathcal{E}$ such that $F\varepsilon_0 = A$ and $F\varepsilon_1 = B$.

So if the exponential $\mathcal{D}^{\mathcal{C}}$ exists, its objects are in 1-to-1 correspondence with functors $1 \to \mathcal{D}^{\mathcal{C}}$, which are in 1-to-1 correspondence with functors $\mathcal{C} \cong \mathcal{C} \times 1 \to \mathcal{D}$. So the objects of the category $\mathcal{D}^{\mathcal{C}}$ must correspond 1-to-1 with functors $\mathcal{C} \to \mathcal{D}$, which means we cannot really go wrong with defining its objects to be functors $\mathcal{C} \to \mathcal{D}$.

Similarly, if F, G are functors $\mathcal{C} \to \mathcal{D}$ then morphisms from F to G in $\mathcal{D}^{\mathcal{C}}$ must correspond 1-to-1 with functors $S : \mathbb{I} \to \mathcal{D}^{\mathcal{C}}$ with $S\varepsilon_0 = F$ and $S\varepsilon_1 = G$. But such functors S correspond to functors $\sigma : \mathcal{C} \times \mathbb{I} \to \mathcal{D}$ with $\sigma(\mathcal{C} \times \varepsilon_0) = F$ and $\sigma(\mathcal{C} \times \varepsilon_1) = G$ (see exercise 3 below). Using the Bifunctor Lemma (see exercise 4 below) one can prove that these correspond to natural transformations $F \Rightarrow G$.

In the remainder of this subsection we will explore some of the functoriality properties of the exponential.

Definition 6.5 An object X in a category $\mathcal{C}$ with finite products is called *exponentiable* if the exponential Y^X exists for each object Y.

Proposition 6.6 *If X is an exponentiable object in a category $\mathcal{C}$, then the operation $Y \mapsto Y^X$ is the object part of an endofunctor $(-)^X$ on $\mathcal{C}$ and* $\mathrm{ev}_Y : Y^X \times X \to Y$ *is the component at Y of a natural transformation* $\mathrm{ev} : (-)^X \times X \Rightarrow 1_{\mathcal{C}}$.

Proof. If $f : Y \to Y'$ is a morphism in $\mathcal{C}$, then the universal property of the exponential $(Y')^X$ tells us that there is a unique map f^X making

$$\begin{tikzcd} Y^X \times X \arrow[r, "\mathrm{ev}_Y"] \arrow[d, "f^X \times 1_X"'] & Y \arrow[d, "f"] \\ (Y')^X \times X \arrow[r, "\mathrm{ev}_{Y'}"] & Y' \end{tikzcd}$$

commute. From the uniquess part of the statement functoriality ($1^X = 1$ and $g^X f^X = (gf)^X$) follows. In addition, the commutativity of the square above tells us that we have defined a natural transformation $\mathrm{ev} : (-)^X \times X \Rightarrow 1_{\mathcal{C}}$. ∎

In a way this is only the beginning: see exercises 5 and 6 below.

6.2 Natural numbers object

Dedekind observed, that in Set, the set $\mathbb{N}$ is characterized by the following property: given any set X, any element $x \in X$ and any function $X \xrightarrow{f} X$, there is a unique function $F: \mathbb{N} \to X$ such that $F(0) = x$ and $F(x+1) = f(F(x))$.

Lawvere took this up, and proposed this *categorical* property as a definition (in a more general context) of a "natural numbers object" in a category.

Definition 6.7 *In a category $\mathcal{C}$ with terminal object 1, a* natural numbers object *is a triple $(0, N, S)$ where N is an object of $\mathcal{C}$ and $1 \xrightarrow{0} N$, $N \xrightarrow{S} N$ arrows in $\mathcal{C}$, such that for any other such diagram*

$$1 \xrightarrow{x} X \xrightarrow{f} X$$

there is a unique *map $\phi: N \to X$ making*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "S"] \arrow[d, "\phi"'] & N \arrow[d, "\phi"] \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

commute.

Of course we think of 0 as the zero element, and of S as the successor map. The defining property of a natural numbers object enables one to "do recursion".

Definition 6.8 *Let $\mathcal{C}$ be a ccc with natural numbers object $(0, N, S)$. We say that a number-theoretic function $F: \mathbb{N}^k \to \mathbb{N}$ is* represented *by an arrow $f: N^k \to N$ if for any k-tuple of natural numbers $n_1, \ldots n_k$, the diagram*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "0"'] & N \arrow[r, "\langle S^{n_1}, \ldots, S^{n_k} \rangle"] & N^k \arrow[d, "f"] \\ & N \arrow[r, "S^{F(n_1, \ldots, n_k)}"'] & N \end{tikzcd}$$

commutes.

The following functions are representable in any ccc with a natural numbers object:

- Addition.
- Multiplication.

- Exponentiation.
- ...

(For this, we refer to the exercises 7 and 8.)

One could ask: what is the class of those numerical functions (that is, functions $\mathbb{N}^k \to \mathbb{N}$) that are representable in every ccc with natural numbers object? The answer is: the representable functions are precisely the so-called ε_0*-recursive functions* from Proof Theory; this is a proper subclass of the set of all computable functions. This was essentially shown by Gödel in 1958.

6.3 Exercises

Exercise 47 Show that in a ccc, there are natural isomorphisms $1^X \cong 1$; $(Y \times Z)^X \cong Y^X \times Z^X$; $(Y^Z)^X \cong Y^{Z\times X}$.

Exercise 48 If a ccc has coproducts, we have $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$ and $Y^{Z+X} \cong Y^Z \times Y^X$.

Exercise 49 In a ccc, prove that the transpose of a composite $Z \xrightarrow{g} W \xrightarrow{f} Y^X$ is

$$Z \times X \xrightarrow{g\times 1_X} W \times X \xrightarrow{\tilde{f}} Y$$

if $\tilde{f}$ is the transpose of f.

Exercise 50 (Bifunctor lemma) Suppose $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ are categories, and we are given:

1. For each pair of objects C in $\mathcal{C}$ and D in $\mathcal{D}$ an object $F_0(C, D)$ in $\mathcal{E}$;
2. For each object $C \in \mathcal{C}$ a functor $F_C: \mathcal{D} \to \mathcal{E}$ satisfying $F_C(D) = F_0(C, D)$ for each object D in $\mathcal{D}$;
3. For each object $D \in \mathcal{D}$ a functor $F_D: \mathcal{C} \to \mathcal{E}$ satisfying $F_D(C) = F_0(C, D)$ for each object C in $\mathcal{C}$;

such that for each pair of morphisms $f: C \to C'$ in $\mathcal{C}$ and $g: D \to D'$ in $\mathcal{D}$ we have a commutative square

$$\begin{tikzcd} F_0(C,D) \arrow[r, "F_D(f)"] \arrow[d, "F_C(g)"] & F_0(C',D) \arrow[d, "F_{C'}(g)"] \\ F_0(C,D') \arrow[r, "F_{D'}(f)"] & F_0(C',D') \end{tikzcd}$$

in $\mathcal{E}$.

Show that there is a unique functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ whose operation on objects is F_0, while $F(1_C, g) = F_C(g)$ and $F(f, 1_D) = F_D(f)$ for each pair of arrows $f: C \to C'$ in $\mathcal{C}$ and $g: D \to D'$ in $\mathcal{D}$.

Exercise 51 An object X in a category $\mathcal{C}$ with finite products is called *exponentiating* if the exponential X^Y exists for each object Y in $\mathcal{C}$. Show that if Y is exponentiating, the assignment $Y \mapsto X^Y$ is the object part of a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$.

Exercise 52 Show that for every ccc $\mathcal{C}$ there is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, assigning Y^X to (X, Y).

Hint: Use the previous two exercises as well as Proposition 5.6.

Exercise 53 Let A be the unique function making

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & \mathbb{N} \arrow[r, "S"] \arrow[d, "A"] & \mathbb{N} \arrow[d, "A"] \\ & \mathbb{N}^{\mathbb{N}} \arrow[r, "S^{\mathbb{N}}"] & \mathbb{N}^{\mathbb{N}} \end{tikzcd}$$

commute. Show that the addition function is represented by the transpose of A.

Hint: Recall that addition is the only function satisfying $0 + n = n$ and $(Sn) + m = S(n + m)$ for all natural numbers n, m.

Exercise 54 Show that multiplication is representable.

Hint: Use the previous exercise both for its result and for inspiration.

7 Presheaves

In category theory an important role is played by the category of presheaves on a small category $\mathcal{C}$.

Definition 7.1 Let $\mathcal{C}$ be a small category. The category of *presheaves* on $\mathcal{C}$ is the functor category

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}],$$

so the category of contravariant functors from $\mathcal{C}$ to Sets (the presheaves on $\mathcal{C}$) and natural transformations between. Other notations that are used for the category of presheaves are $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\widehat{\mathcal{C}}$.

We will see in this and the next section that a category of presheaves $\widehat{\mathcal{C}}$ has many properties that make it similar to the category of sets (it is cartesian closed and both complete and cocomplete, for instance); moreover, the category $\mathcal{C}$ embeds into $\widehat{\mathcal{C}}$ via the important Yoneda embedding. This will prove, in particular, that any small category embeds into one which is both complete and cocomplete.

7.1 Examples and first properties

Before we delve into the properties of a category of presheaves, let us first point out that you are probably already aware of a few examples.

Examples of presheaf categories

1. A first example is the category of presheaves on a monoid (a one-object category) M. Such a presheaf is nothing but a set X together with a right M-*action*, that is: we have a map $X \times M \to X$, written $x, f \mapsto xf$, satisfying $xe = x$ (for the unit e of the monoid), and $(xf)g = x(fg)$. There is only one representable presheaf.

2. The category of directed graphs and graph morphisms is a presheaf category: it is the category of presheaves on the category with two objects e and v, and two non-identity arrows $\sigma, \tau\colon v \to e$. For a presheaf X on this category, $X(v)$ can be seen as the set of vertices, $X(e)$ the set of edges, and $X(\sigma), X(\tau)\colon X(e) \to X(v)$ as the source and target maps. (Note that we allow for parallel edges in this definition of a directed graph.)

3. A *forest* is a partially ordered set such that for any $x \in T$, the set $\downarrow(x) = \{y \in T \mid y \leq x\}$ is a finite linearly ordered subset of T. A *morphism of forests* $f\colon T \to S$ is an order-preserving function wth the property that for any element $x \in T$, the restriction of f to $\downarrow(x)$ is a bijection from $\downarrow(x)$ to $\downarrow(f(x))$. A *tree* is a forest with a least element; a morphism of trees is just

a morphism of forests. The category of forests and trees are isomorphic and both are isomorphic to the category of presheaves on $\mathbb{N}$ with the usual poset structure considered as a category.

We have seen that a functor category of the form $[\mathcal{C}, \mathcal{D}]$ has whatever limits or colimits $\mathcal{D}$ has. Indeed, since Sets is both complete and cocomplete, we have:

Corollary 7.2 *The category* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *is both complete and cocomplete, with limits and colimits calculated "pointwise".*

So, for instance, the initial object of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the constant presheaf with value $\emptyset$ and X is terminal in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ if and only if every set $X(C)$ is a singleton.

In addition, we have the following result:

Proposition 7.3 *Assume $\mathcal{D}$ is a category with pullbacks. Then a natural transformation σ in $[\mathcal{C}, \mathcal{D}]$ is a mono if and only if each component σ_C is monic in $\mathcal{D}$. Dually, if $\mathcal{D}$ has pushouts, a natural transformation in $[\mathcal{C}, \mathcal{D}]$ will be an epi if and only if each component is epi.*

Proof. This follows from two observations. First, if $\mathcal{D}$ has pullbacks, then pullbacks in $[\mathcal{C}, \mathcal{D}]$ are computed pointwise: in other words, a square in $[\mathcal{C}, \mathcal{D}]$ will be a pullback if and only if it is a pointwise pullback. Second, in a category a map $f\colon A \to B$ is monic if and only if

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback. ∎

For categories of presheaves this means that a map of presheaves is an epi if and only if it is pointwise surjective, and a mono if and only if it is pointwise injective. To uncover more structure of the category of presheaves, we need the Yoneda embedding.

7.2 The Yoneda Lemma

For each locally small category $\mathcal{C}$ there is a functor

$$\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sets},$$

which sends a pair (A, B) of objects in $\mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(A, B)$; in addition, if (f, g) is a morphism $(A, B) \to (A', B')$ in $\mathcal{C}$ (so $f\colon A' \to A$ in $\mathcal{C}$ and $g\colon B \to B'$ in $\mathcal{D}$), then there is an operation

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A', B')$$

sending $h: A \to B$ to $g \circ h \circ f: A' \to B'$. Functoriality is easily verified.

Since $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is an exponential, the functor $\mathrm{Hom}_{\mathcal{C}}$ corresponds to a functor

$$y: \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}.$$

To spell out what this means, we have for each object C in $\mathcal{C}$ a presheaf y_C with $(y_C)(D) = \mathrm{Hom}_{\mathcal{C}}(D, C)$ for each object D in $\mathcal{C}$ and for each $f: D \to D'$ an operation $(y_C)(D') \to (y_C)(D)$ given by precomposition with f. In addition, if $g: C \to C'$ is a morphism in $\mathcal{C}$ we get a natural transformation $y_g: y_C \to y_{C'}$ whose component

$$(y_g)_D: (y_C)(D) \to (y_{C'})(D)$$

at D is given by postcomposition with g. The functor y is called the *Yoneda embedding*. An embedding is a functor which is full and faithful and injective on objects. To prove that y is an embedding, we will need the following result, which is one of the most important facts in category theory.

Theorem 7.4 (Yoneda Lemma) *For every object F of* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *and every object C of $\mathcal{C}$, the operation*

$$\alpha_{C,F}: \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, F) \to F(C)$$

sending a natural transformation $\sigma: y_C \to F$ to $(\sigma_C)(1_C)$ is a bijection. Moreover, this bijection is natural *in C and F in the following sense: given $g: C' \to C$ in $\mathcal{C}$ and $\mu: F \Rightarrow F'$ in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, *the diagram*

$$\begin{tikzcd}
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, F) \arrow[r, "f_{C,F}"] \arrow[d, "\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(g,\mu)"'] & F(C) \arrow[d, "\mu_{C'}F(g)=F'(g)\mu_C"] \\
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_{C'}, F') \arrow[r, "f_{C',F'}"'] & F'(C')
\end{tikzcd}$$

commutes in Set. *In other words, α is a natural isomorphism between the functors*

$$\mathcal{C}^{\mathrm{op}} \times \widehat{\mathcal{C}} \xrightarrow{\mathrm{ev}} \mathrm{Sets}$$

and

$$\mathcal{C}^{\mathrm{op}} \times \widehat{\mathcal{C}} \xrightarrow{y \times 1} \widehat{\mathcal{C}}^{\mathrm{op}} \times \widehat{\mathcal{C}} \xrightarrow{\mathrm{Hom}} \mathrm{Sets}.$$

Proof. If $\kappa = (\kappa_{C'} | C' \in \mathcal{C}_0)$ is a natural transformation: $y_C \Rightarrow F$ then, $\kappa_{C'}(f)$ must be equal to $F(f)(\kappa_C(1_C))$. So κ is completely determined by $\kappa_C(1_C) \in F(C)$ and conversely, any element of $F(C)$ determines a natural transformation $y_C \Rightarrow F$.

Given $g: C' \to C$ in $\mathcal{C}$ and $\mu: F \Rightarrow F'$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, the map $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(g, \mu)$ sends the natural transformation $\kappa = (\kappa_{C''} | C'' \in \mathcal{C}_0): y_C \Rightarrow F$ to $\lambda = (\lambda_{C''} | C'' \in \mathcal{C}_0)$ where $\lambda_{C''}: y_{C'}(C'') \to F'(C'')$ is defined by

$$\lambda_{C''}(h: C'' \to C') = \mu_{C''}(\kappa_{C''}(gh))$$

Now

$$\begin{array}{rcl} f_{C',F'}(\lambda) & = & \lambda_{C'}(1_{C'}) \\ & = & \mu_{C'}(\kappa_{C'}(g)) \\ & = & \mu_{C'}(F(g)(\kappa_C(1_C))) \\ & = & (\mu_{C'} F(g))(f_{C,F}(\kappa)) \end{array}$$

which proves the naturality statement. ∎

Definition 7.5 A presheaf which is isomorphic to one of the form y_C is called *representable*.

Let X be a presheaf on $\mathcal{C}$, and let $y{\downarrow}X$ be the following category (this is an example of a 'comma category' construction): objects are pairs (C, μ) with C an object of $\mathcal{C}$ and $\mu: y_C \to X$ an arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. A morphism $(C, \mu) \to (C', \nu)$ is an arrow $f: C \to C'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} y_C \arrow[rr, "y_f"] \arrow[dr, "\mu"'] & & y_{C'} \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

commutes.

Note that if this is the case and $\mu: y_C \to X$ corresponds to $\xi \in X(C)$ and $\nu: y_{C'} \to X$ corresponds to $\eta \in X(C')$, then $\xi = X(f)(\eta)$.

There is a functor $U_X: y{\downarrow}X \to \mathcal{C}$ (the forgetful functor) which sends (C, μ) to C and f to itself; by composition with y we get a diagram

$$y{\circ}U_X: y{\downarrow}X \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

Clearly, there is a natural transformation ρ from $y{\circ}U_X$ to the constant functor Δ_X from $y{\downarrow}X$ to $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ with value X: let $\rho_{(C,\mu)} = \mu: y_C \to X$. So there is a cocone in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ for $y{\circ}U_X$ with vertex X.

Proposition 7.6 *The cocone $\rho: y{\circ}U_X \Rightarrow \Delta_X$ is colimiting.*

Proof. Suppose $\lambda: y{\circ}U_X \Rightarrow \Delta_Y$ is another cocone. Define $\nu: X \to Y$ by $\nu_C(\xi) = (\lambda_{(C,\mu)})_C(\mathrm{id}_C)$, where $\mu: y_C \to X$ corresponds to ξ in the Yoneda Lemma.

Then ν is natural: if $f{:}\, C' \to C$ in $\mathcal{C}$ and $\mu'{:}\, y_{C'} \to X$ corresponds to $X(f)(\xi)$, the diagram

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\mu'"'] & & y_C \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

commutes, so f is an arrow $(C', \mu') \to (C, \mu)$ in $y{\downarrow}X$. Since λ is a cocone, we have that

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\lambda_{(C',\mu')}"'] & & y_C \arrow[dl, "\lambda_{(C,\mu)}"] \\ & Y & \end{tikzcd}$$

commutes; so

$$\begin{array}{lcrl} \nu_{C'}(X(f)(\xi)) & = & (\lambda_{(C',\mu')})_{C'}(\mathrm{id}_{C'}) & = \\ (\lambda_{(C,\mu)})_{C'}((y_f)_{C'}(\mathrm{id}_{C'})) & = & (\lambda_{(C,\mu)})_{C'}(f) & = \\ Y(f)((\lambda_{(C,\mu)})_C(\mathrm{id}_C)) & = & Y(f)(\nu_C(\xi)) & \end{array}$$

It is easy to see that $\lambda{:}\, y{\circ}U_X \Rightarrow \Delta_Y$ factors through ρ via ν, and that the factorization is unique. ∎

Proposition 7.6 is often referred to by saying that "every presheaf is a colimit of representables".

Furthermore we note the following fact: the Yoneda embedding $\mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the 'free colimit completion' of $\mathcal{C}$. That is: for any functor $F{:}\, \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a cocomplete category, there is, up to isomorphism, exactly one *colimit preserving* functor $\tilde{F}{:}\, \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\ & \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

commutes. $\tilde{F}(X)$ is computed as the colimit in $\mathcal{D}$ of the diagram

$$y{\downarrow}X \xrightarrow{U_X} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

The functor $\tilde{F}$ is also called the 'left Kan extension of F along y'.

7.3 Applications of the Yoneda Lemma

In the remainder of the course will see numerous applications of the Yoneda Lemma. We can already mention the following three.

7.3.1 The Yoneda embedding

First of all, the Yoneda Lemma can be used to see that y is indeed an embedding (full and faithful and injective on objects). That y is injective on objects is easy to see, because $1_C \in y_C(C)$ for each object C, and 1_C is in no other set $y_D(E)$. In addition:

Corollary 7.7 *The functor $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is full and faithful.*

Proof. Immediate by the Yoneda lemma, since

$$\mathcal{C}(C, C') = y_{C'}(C) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, y_{C'})$$

and this bijection is induced by y. ∎

7.3.2 Arguments from representability

Another typical application of the Yoneda lemma is the following. One wants to prove that objects A and B of $\mathcal{C}$ are isomorphic. Suppose one can show that for every object X of $\mathcal{C}$ there is a bijection $f_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ which is natural in X; i.e. given $g : X' \to X$ in $\mathcal{C}$ one has that

$$\begin{tikzcd}
\mathcal{C}(X, A) \arrow[r, "f_X"] \arrow[d, "\mathcal{C}(g, 1_A)"'] & \mathcal{C}(X, B) \arrow[d, "\mathcal{C}(g, 1_B)"] \\
\mathcal{C}(X', A) \arrow[r, "f_{X'}"'] & \mathcal{C}(X', B)
\end{tikzcd}$$

commutes.

Then one can conclude that A and B are isomorphic in $\mathcal{C}$; for, from what one has just shown it follows that y_A and y_B are isomorphic objects in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Since y is full and faithful, this map is of the form y_f for some isomorphism f in $\mathcal{C}$.

7.3.3 Presheaves are cartesian closed

In addition, we can show that category of presheaves is cartesian closed. Indeed, exponentials can be calculated using the Yoneda Lemma. For Y^X, we need a natural 1-1 correspondence

$$\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(Z, Y^X) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(Z \times X, Y)$$

In particular this should hold for representable presheaves y_C; so, by the Yoneda Lemma, we should have a 1-1 correspondence

$$Y^X(C) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$$

which is natural in C. This leads us to *define* a presheaf Y^X by:

$$Y^X(C) = \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y),$$

and for $f: C' \to C$ we let $Y^X(f): Y^X(C) \to Y^X(C')$ be defined by composition with $y_f \times \mathrm{id}_X: y_{C'} \times X \to y_C \times X$. Then certainly, Y^X is a well-defined presheaf. One can now check (exercise!) that together with the *evaluation map* $\mathrm{ev}_{X,Y}: Y^X \times X \to Y$ given by

$$(\phi, x) \;\mapsto\; \phi_C(\mathrm{id}_C, x)$$

the presheaf Y^X is the exponential in the category of presheaves.

7.4 Exercises

Exercise 55 Suppose objects A and B are such that for every object X in $\mathcal{C}$ there is a bijection $f_X: \mathcal{C}(A, X) \to \mathcal{C}(B, X)$, naturally in a sense you define yourself. Conclude that A and B are isomorphic (hint: duality!).

Exercise 56 Show that the following are equivalent for each small category $\mathcal{C}$:

(1) $\mathcal{C}$ has a terminal object.

(2) The terminal object in $\widehat{\mathcal{C}}$ is representable.

Exercise 57 Show that the following are equivalent for each small category $\mathcal{C}$:

(1) $\mathcal{C}$ has binary products.

(2) For each pair of objects A and B in $\mathcal{C}$ the presheaf $yA \times yB$ is representable in $\widehat{\mathcal{C}}$.

Can you generalise the statement in this and previous exercise to general limits?

Exercise 58 Again, let $\mathcal{C}$ be a small category with binary products, and let A and B be objects in $\mathcal{C}$.

(a) Show that the assignment

$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(X \times A, B)$$

is part of a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$, with the action on morphisms $f: X' \to X$ in $\mathcal{C}$ given by precomposition with $f \times 1_A$.

(b) What does it say about $\mathcal{C}$ if the functor in part (a) is representable?

Exercise 59 Prove that $y: \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. Prove also, that if $\mathcal{C}$ is cartesian closed, y preserves exponents.

8 Presheaves as a topos

The aim of this section is to show that presheaves on a small category form a topos.

Definition 8.1 An *(elementary) topos* is a category with finite limits, which is cartesian closed and has a subobject classifier.

So what we have to explain is what a subobject classifier is and why the category of presheaves has such a subobject classifier.

8.1 Subobject classifier

Another piece of structure we shall need is that of a *subobject classifier*.

Definition 8.2 Suppose $\mathcal{E}$ is a category with finite limits. A *subobject classifier* is a monomorphism $t{:}\,T \to \Omega$ with the property that for any monomorphism $m{:}\,A \to B$ in $\mathcal{E}$ there is a *unique* arrow $\phi{:}\,B \to \Omega$ such that there is a pullback diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "m"'] & T \arrow[d, "t"] \\ B \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

(Note that there is at most one map ψ making the square commute, because t is monic.) We say that the unique arrow ϕ *classifies* m or rather, the subobject represented by m (if m and m' represent the same subobject, they have the same classifying arrow).

In Set, the two element set $\{\top, \bot\}$ together with the arrow $1 \to \{\top, \bot\}$ picking out $\top$ acts as a subobject classifier: for $A \subseteq B$ we have the unique *characteristic function* $\phi_A{:}\,B \to \{\top, \bot\}$ defined by $\phi_A(x) = \top$ if $x \in A$, and $\phi_A(x) = \bot$ otherwise.

It is no coincidence that in Set, the domain of $t{:}\,T \to \Omega$ is a terminal object: T is always terminal.

Lemma 8.3 *Let $\mathcal{E}$ be a category with finite limits and $t{:}\,T \to \Omega$ be a subobject classifier. Then T is terminal.*

Proof. For, for any object A the arrow $\phi{:}\,A \to \Omega$ which classifies the identity on A factors as tn for some $n{:}\,A \to T$. On the other hand, if $k{:}\,A \to T$ is any arrow, then we have pullback diagrams

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "\mathrm{id}_A"'] & T \arrow[r, "\mathrm{id}_T"] \arrow[d, "\mathrm{id}_T"] & T \arrow[d, "t"] \\ A \arrow[r, "k"'] & T \arrow[r, "t"'] & \Omega \end{tikzcd}$$

so tk classifies id_A. By uniqueness of the classifying map, $tn = tk$; since t is mono, $n = k$. So T is terminal. ∎

Henceforth we shall write $1 \xrightarrow{t} \Omega$ for the subobject classifier, or, by abuse of language, just Ω.

To explain the name subobject classifier, let us explain what subobjects are.

Definition 8.4 Let X be an object in some category $\mathcal{E}$. The monos with codomain X can be preordered by saying that $(m: A \to X) \leq (n: B \to X)$ holds whenever there is a (necessarily unique and monic) map $k: A \to B$ such that $nk = m$. An isomorphism class of monos in this preorder is called a *subobject* on X, and we write

$$\mathrm{Sub}_{\mathcal{E}}(X)$$

for the "poclass" (partially ordered class) of subobjects on X. If each $\mathrm{Sub}_{\mathcal{E}}(X)$ is a set, we call $\mathcal{E}$ *well-powered*.

Note that in the category of Sets there is an order-preserving bijection between the subobjects on X and the subsets of X. Indeed, if $A \subseteq X$, then this gives rise to an inclusion map $i: A \to X$; conversely, if $m: A \to X$ is injective, then it can be written as

$$A \cong \mathrm{Im}(m) \subseteq X.$$

Since the pullback of a mono is again a mono, we have for each morphism $f: Y \to X$ in $\mathcal{E}$ a *pullback functor* (monotone function)

$$f^*: \mathrm{Sub}_{\mathcal{E}}(X) \to \mathrm{Sub}_{\mathcal{E}}(Y).$$

Indeed, if $\mathcal{E}$ is small, we could regard $\mathrm{Sub}_{\mathcal{E}}$ as a functor

$$\mathcal{E}^{\mathrm{op}} \to \mathcal{S}ets.$$

Having a subobject classifier is equivalent to saying that that this functor is representable. In other words, having a subobject classifier means that there is an object Ω such that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(X, \Omega) \cong \mathrm{Sub}_{\mathcal{E}}(X).$$

Indeed, the Yoneda Lemma tells us that this statement is equivalent to there being a special subobject, represented by $t: T \to \Omega$, say, such that any other subobject is a pullback of that one.

This leads us to consider more general *power objects*. In a category $\mathcal{E}$ with finite products, we call an object A a *power object* of the object X, if there is a natural 1-1 correspondence

$$\mathcal{E}(Y, A) \cong \mathrm{Sub}_{\mathcal{E}}(X \times Y)$$

The naturality means that if $f{:}\, Y \to A$ and $g{:}\, Z \to Y$ are arrows in $\mathcal{E}$ and f corresponds to the subobject U of $X \times Y$, then $fg{:}\, Z \to A$ corresponds to the subobject $(\mathrm{id}_X \times g)^*(U)$ of $X \times Z$.

Power objects are unique up to isomorphism; the power object of X, if it exists, is usually denoted $\mathcal{P}(X)$. Note the following consequence of the definition: to the identity map on $\mathcal{P}(X)$ corresponds a subobject of $X \times \mathcal{P}(X)$ which we call the "element relation" $\in_X$. So, again by Yoneda, a power object $\mathcal{P}(X)$ on X can be defined as an object equipped with subobject of $\in_X$ of $X \times \mathcal{P}(X)$ such that any subobject U of $X \times Y$ is of the form $U = (\mathrm{id}_X \times f)^*(\in_X)$ for a unique map $f{:}\, Y \to \mathcal{P}(X)$.

Please convince yourself that power objects in the category Set are just the familiar power sets.

8.2 Subobject classifiers in presheaves

To compute subobject classifiers and power objects in presheaves, let us first discuss subobjects in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. A subobject of X can be identified with a *subpresheaf* of X: that is, a presheaf Y such that $Y(C) \subseteq X(C)$ for each C, and $Y(f)$ is the restriction of $X(f)$ to $Y(\mathrm{cod}(f))$. This follows easily from the corresponding fact in Set.

Again, we use the Yoneda Lemma to compute the subobject classifier in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. We need a presheaf Ω such that at least for each representable presheaf y_C, $\Omega(C)$ is in 1-1 correspondence with the set of subobjects (in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$) of y_C. So we define Ω such that $\Omega(C)$ is the set of subpresheaves of y_C; for $f{:}\, C' \to C$ we have $\Omega(f)$ defined by the action of pulling back along y_f.

What do subpresheaves of y_C look like? If R is a subpresheaf of y_C then R can be seen as a set of arrows with codomain C such that if $f{:}\, C' \to C$ is in R and $g{:}\, C'' \to C'$ is arbitrary, then fg is in R (for, $fg = y_C(g)(f)$). Such a set of arrows is called a *sieve* on C.

Under the correspondence between subobjects of y_C and sieves on C, the operation of pulling back a subobject along a map y_f (for $f{:}\, C' \to C$) sends a sieve R on C to the sieve $f^*(R)$ on C' defined by

$$f^*(R) \;=\; \{g{:}\, D \to C' \mid fg \in R\}$$

So Ω can be defined as follows: $\Omega(C)$ is the set of sieves on C, and $\Omega(f)(R) = f^*(R)$. The map $t{:}\, 1 \to \Omega$ sends, for each C, the unique element of $1(C)$ to the maximal sieve on C (i.e., the unique sieve which contains id_C).

Let us now prove that $t: 1 \to \Omega$, thus defined, is a subobject classifier in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Let Y be a subpresheaf of X. Then for any C and any $x \in X(C)$, the set

$$\phi_C(x) \ = \ \{f: D \to C \mid X(f)(x) \in Y(D)\}$$

is a sieve on C, and defining $\phi: X \to \Omega$ in this way gives a natural transformation: for $f: C' \to C$ we have

$$\begin{array}{rcr} \phi_{C'}(X(f)(x)) & = & \{g: D \to C' \mid X(g)(X(f)(x)) \in Y(D)\} \\ & = & \{g: D \to C' \mid X(gf)(x) \in Y(D)\} \\ & = & \{g: D \to C' \mid fg \in \phi_C(x)\} \\ & = & f^*(\phi_C(x)) \\ & = & \Omega(f)(\phi_C(x)) \end{array}$$

Moreover, if we take the pullback of t along ϕ, we get the subpresheaf of X consisting of (at each object C) of those elements x for which $\mathrm{id}_C \in \phi_C(x)$; that is, we get Y. So ϕ classifies the subpresheaf Y.

On the other hand, if $\phi: X \to \Omega$ is any natural transformation such that pulling back t along ϕ gives Y, then for every $x \in X(C)$ we have that $x \in Y(C)$ if and only if $\mathrm{id}_C \in \phi_C(x)$. But then by naturality we get for any $f: C' \to C$ that

$$f \in \phi_C(x) \ \Leftrightarrow \ \mathrm{id}_{C'} \in f^*(\phi_C(x)) \ \Leftrightarrow \ X(f)(x) \in Y(C')$$

which shows that the classifying map ϕ is unique.

So this shows that the category of presheaves is a topos. This also shows that the category of presheaves has power objects, because in a topos $\mathcal{E}$ we can put $\mathcal{P}(X) = \Omega^X$. Indeed, for any topos $\mathcal{E}$ we have

$$\mathrm{Hom}_{\mathcal{E}}(Y, \Omega^X) \cong \mathrm{Hom}_{\mathcal{E}}(X \times Y, \Omega) \cong \mathrm{Sub}_{\mathcal{E}}(X \times Y),$$

naturally in X. An explicit description of the power objects in presheaves can be found in the exercises below.

8.3 Exercises

Exercise 60 Suppose $\mathcal{C}$ is a preorder $(P, \leq)$. For $p \in P$ we let $\downarrow(p) = \{q \in P \mid q \leq p\}$. Show that sieves on p can be identified with downwards closed subsets of $\downarrow(p)$. If we denote the unique arrow $q \to p$ by qp and U is a downwards closed subset of $\downarrow(p)$, what is $(qp)^*(U)$?

Exercise 61 Let $\mathcal{C}$ be a small category. Show that the power object in the category of presheaves on $\mathcal{C}$ can be defined as follows: $\mathcal{P}(X)(C) \ = \ \mathrm{Sub}(X \times y_C)$ and that, for $f: C' \to C$, $\mathcal{P}(X)(f)(U) = (\mathrm{id}_X \times y_f)^*(U)$. Prove also, that the element relation, as a subpresheaf $\in_X$ of $\mathcal{P}(X) \times X$, is given by

$$(\in_X)(C) \ = \ \{(U, x) \in \mathrm{Sub}(y_C \times X) \times X(C) \mid (\mathrm{id}_C, x) \in U(C)\}.$$

Exercise 62 Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$. Let P be such a presheaf. We define a presheaf $\tilde{P}$ as follows: for an object C of $\mathcal{C}$, $\tilde{P}(C)$ consists of those subobjects α of $y_C \times P$ which satisfy the following condition: for all arrows $f\colon D \to C$, the set

$$\{y \in P(D) \mid (f, y) \in \alpha(D)\}$$

has at most one element.

a) Complete the definition of $\tilde{P}$ as a presheaf.

b) Show that there is a monic map $\eta_P\colon P \to \tilde{P}$ with the following property: for every diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "m"'] & P \\ B & \end{tikzcd}$$

with m mono, there is a unique map $\tilde{g}\colon B \to \tilde{P}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "m"'] & P \arrow[d, "\eta_P"] \\ B \arrow[r, "\tilde{g}"'] & \tilde{P} \end{tikzcd}$$

is a pullback square. The arrow $P \xrightarrow{\eta_P} \tilde{P}$ is called a *partial map classifier* for P.

c) Show that the assignment $P \mapsto \tilde{P}$ is part of a functor $\tilde{(\cdot)}$ in such a way that the maps η_P form a natural transformation from the identity functor to $\tilde{(\cdot)}$, and all naturality squares for η are pullbacks.

Exercise 63 Let $\mathcal{E}$ be a topos with subobject classifier $1 \xrightarrow{t} \Omega$.

a) Prove that Ω is injective. (An object I is *injective* if for any pair of maps $f\colon A \to I$ and $g\colon A \to B$ with g monic, there is a map $h\colon B \to I$ such that $hg = f$; the notion of an injective object is dual to that of a projective object.)

b) Prove that every object of the form Ω^X is injective.

c) Conclude that $\mathcal{E}$ has enough injectives. (A category $\mathcal{E}$ *has enough injectives* if for any object X there is an injective object I and a monomorphism $m\colon X \to I$.)

9 Adjunctions

The following kind of problem occurs quite regularly: suppose we have a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ and for a given object C of $\mathcal{C}$, we look for an object $G(D)$ which "best approximates" C, in the sense that there is a map $C \xrightarrow{\eta} G(D)$ with the property that any other map $C \xrightarrow{g} G(D')$ factors uniquely as $G(f)\eta$ for $f\colon D \to D'$ in $\mathcal{D}$.

We have seen, that if G is the inclusion of Abgp into Grp, the abelianization of a group is an example. Another example is the Čech-Stone compactification in topology: for a completely regular topological space X one constructs a compact Hausdorff space βX out of it, and a map $X \to \beta X$, such that any continuous map from X to a compact Hausdorff space factors uniquely through this map.

Of course, what we described here is a sort of "right-sided" approximation; the reader can define for himself what the notion for a left-sided approximation would be.

The categorical description of this kind of phenomena goes via the concept of *adjunction*, which this chapter is about.

9.1 Definition and examples

Definition 9.1 Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of functors between categories $\mathcal{C}$ and $\mathcal{D}$. An *adjunction* between F and G is a natural isomorphism between the two functors

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}ets$$

given by $\mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times 1_{\mathcal{D}})$ and $\mathrm{Hom}_{\mathcal{C}} \circ (1_{\mathcal{C}^{\mathrm{op}}} \times G)$. If such an adjunction exists, we say that F is *left adjoint* to G, or G is *right adjoint* to F. We write this as $F \dashv G$.

So an adjunction is a natural bijection:

$$\mathcal{D}(FC, D) \xrightarrow{m_{C,D}} \mathcal{C}(C, GD)$$

for each pair of objects $C \in \mathcal{C}_0, D \in \mathcal{D}_0$. Two maps $f\colon FD \to C$ in $\mathcal{C}$ and $g\colon D \to GC$ in $\mathcal{D}$ which correspond to each other under this correspondence are called *transposes* of each other. If m is understood, we will often write $\overline{f}$ and $\overline{g}$ for the transposes of f and g, respectively (so $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$).

The naturality of $m_{C,D}$ means that, given $f\colon C' \to C$, $g\colon D \to D'$ in $\mathcal{C}$ and $\mathcal{D}$ respectively, the diagram

$$\begin{array}{ccc}\mathcal{D}(FC,D) & \xrightarrow{m_{C,D}} & \mathcal{C}(C,GD)\\ \Big\downarrow{\scriptstyle \mathcal{D}(Ff,g)} & & \Big\downarrow{\scriptstyle \mathcal{C}(f,Gg)}\\ \mathcal{D}(FC',D') & \xrightarrow{m_{C',D'}} & \mathcal{C}(C',GD')\end{array}$$

commutes in Set. In other words, for $\alpha: FC \to D$ in $\mathcal{D}$ we must have

$$Gg \circ \overline{\alpha} \circ f = \overline{g \circ \alpha \circ Ff}.$$

This single equation is equivalent to the special cases where $f = 1$ and $g = 1$:

$$Gg \circ \overline{\alpha} = \overline{g \circ \alpha} \quad \text{ and } \quad \overline{\alpha} \circ f = \overline{\alpha \circ Ff}.$$

Of course, the naturality of $m_{C,D}$ is equivalent to the naturality of $m^{-1}_{C,D}$, which we can state as the following equation:

$$g \circ \overline{\beta} \circ Ff = \overline{Gg \circ \beta \circ f}$$

where $\beta: C \to GD$. This equation can again be split up in

$$g \circ \overline{\beta} = \overline{Gg \circ \beta} \quad \text{ and } \overline{\beta} \circ Ff = \overline{\beta \circ f}.$$

Examples. The world is full of examples of adjoint functors. We have met several:

a) Consider the forgetful functor U: Grp $\to$ Set and the free functor F: Set $\to$ Grp. Given a function from a set A to a group G (which is an arrow $A \to U(G)$ in Set) we can uniquely extend it to a group homomorphism from $(\tilde{A}, \star)$ to G (see example d) of 1.1), i.e. an arrow $F(A) \to G$ in Grp, and conversely. This is natural in A and G, so $F \dashv U$;

b) The functor Dgrph $\to$ Cat given in example b) of 1.1 is left adjoint to the forgetful functor Cat $\to$ Dgrph;

c) Given functors $P \underset{G}{\overset{F}{\rightleftarrows}} Q^{\mathrm{op}}$ between two preorders P and Q, $F \dashv G$ if and only if we have the equivalence

$$x \leq G(y) \Leftrightarrow y \leq F(x)$$

for $x \in P, y \in Q$; in order theory such a situation is called a *Galois connection*.

d) In Exercise 6 of Chapter 1 we did "abelianization" of a group G. We made use of the fact that any homomorphism $G \to H$ with H abelian, factors uniquely through $G/[G,G]$, giving a natural 1-1 correspondence

$$\mathrm{Grp}(G, I(H)) \xrightarrow{\sim} \mathrm{Abgp}(G/[G,G], H)$$

where I: Abgp $\to$ Grp is the inclusion. So abelianization is left adjoint to the inclusion functor of abelian groups into groups;

e) The *free monoid* $F(A)$ on a set A is just the set of strings on the alphabet A. $F\colon \mathrm{Set} \to \mathrm{Mon}$ is a functor left adjoint to the forgetful functor from Mon to Set;

f) If X is an object in a cartesian closed category $\mathcal{C}$, then the functor $(-) \times X\colon \mathcal{C} \to \mathcal{C}$ is left adjoint to the functor $(-)^X\colon \mathcal{C} \to \mathcal{C}$.

g) Exercise 6 of Chapter 2 gives two functors $F, G\colon \mathrm{Set} \to \mathrm{Cat}$, assigning to a set the discrete and indiscrete category of that set. F and G are respectively left and right adjoint to the functor $\mathrm{Cat} \stackrel{\mathrm{Ob}}{\to} \mathrm{Set}$ which assigns to a (small) category its set of objects (to be precise, for this example to work we have to take for Cat the category of *small* categories), and to a functor its action on objects.

h) If $U\colon \mathrm{Top} \to \mathrm{Set}$ is the forgetful functor which assigns to a topological space its set of points, the functors assigning to a set X the discrete and indiscrete topology on that set are its left and right adjoint, respectively.

i) If $f\colon Y \to X$ is a morphism in a category $\mathcal{C}$, then precomposition with f defines a functor $f_!\colon \mathcal{C}/Y \to \mathcal{C}/X$. If $\mathcal{C}$ has pullbacks, then this functor has a right adjoint $f^*\colon \mathcal{C}/X \to \mathcal{C}/Y$ which assigns to each morphism $g\colon A \to X$ its pullback:

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d, "f^*g"] & A \arrow[d, "g"] \\
Y \arrow[r, "f"] & X.
\end{tikzcd}$$

The reader is encouraged to check that this is indeed a functor whose action on a morphism $k\colon g \to h$ in $\mathcal{C}/X$ is the unique dotted arrow making

$$\begin{tikzcd}
f^*B \arrow[rrr] \arrow[dr, dotted] \arrow[ddr, bend right, "f^*h"'] & & & B \arrow[dl, "k"'] \arrow[ddl, bend left, "h"] \\
& f^*A \arrow[r] \arrow[d, "f^*g"'] & A \arrow[d, "g"'] & \\
& Y \arrow[r, "f"] & X &
\end{tikzcd}$$

commute.

j) Let $P\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ be the functor which takes the powerset on objects, and for $X \stackrel{f}{\to} Y$, $P(f)\colon P(Y) \to P(X)$ gives for each subset B of Y its inverse image under f.

Now P might as well be regarded as a functor $\mathrm{Set} \to \mathrm{Set}^{\mathrm{op}}$; let's write $\bar{P}$ for that functor. Since there is a natural bijection:

$$\mathrm{Set}(X, P(Y)) \stackrel{\sim}{\to} \mathrm{Set}(Y, P(X)) = \mathrm{Set}^{\mathrm{op}}(\bar{P}(X), Y)$$

we have an adjunction $\bar{P} \dashv P$.

9.2 Unit and counit of an adjunction

Suppose we have an adjoint pair of functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ with a natural bijective correspondence

$$\mathcal{D}(FC, D) \xrightarrow{m_{C,D}} \mathcal{C}(C, GD) \cdot$$

If we fix D, then naturality says that the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}: C \mapsto \mathcal{D}(FC, D)$$

is representable and represented by GD. The Yoneda Lemma tells us that for $\varepsilon_D = m^{-1}_{GD,D}(1_{GD}): FGD \to D$ in $\mathcal{D}$ we have

$$m^{-1}_{C,D}(\beta) = \varepsilon_D \circ F\beta.$$

Dually, there is for each object C in $\mathcal{C}$ a map $\eta_C = m_{C,FC}(1_{FC}): C \to GFC$ such that

$$m_{C,D}(\alpha) = G\alpha \circ \eta_C.$$

(Indeed, if $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is an adjunction with F left adjoint to G, then

$$\mathcal{C}^{\mathrm{op}} \underset{F^{\mathrm{op}}}{\overset{G^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\mathrm{op}}$$

is an adjunction too, with G^{op} left adjoint to F^{op}.)

Proposition 9.2 *The maps $(\eta_C : C \in \mathcal{C}_0)$ form a natural transformation $1_{\mathcal{C}} \Rightarrow GF$, while the maps $(\varepsilon_D : D \in \mathcal{D}_0)$ form a natural transformation $FG \Rightarrow 1_{\mathcal{D}}$.*

Proof. By duality, we only need to prove the first statement. So let $f: C' \to C$ and note that

$$GFf \circ \eta_{C'} = \overline{Ff} = \overline{1_{FC} \circ Ff} = \overline{1_{FC}} \circ f = \eta_C \circ f,$$

expresses that η is natural. ∎

We call the natural transformation $\eta: 1_{\mathcal{C}} \Rightarrow GF$ the *unit* and the natural transformation $\varepsilon: FG \Rightarrow 1_{\mathcal{D}}$ the *counit* of the adjunction.

The fact that $m_{C,D}$ and $m^{-1}_{C,D}$ are each others inverses amounts to saying that for all $\alpha: FC \to D$ and $\beta: C \to GD$ the diagrams

$$\begin{tikzcd} C \arrow[r, "\beta"] \arrow[d, "\eta_C"'] & GD \\ GFC \arrow[r, "GF(\beta)"'] & GFG(D) \arrow[u, "G(\varepsilon_D)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FC \arrow[r, "\alpha"] \arrow[d, "F(\eta_C)"'] & D \\ FGFC \arrow[r, "FG(\alpha)"'] & FGD \arrow[u, "\varepsilon_D"'] \end{tikzcd}$$

commute. Because η and ε are natural transformations, these squares will commute as soon as they commute for $\alpha = 1_{FC}$ and $\beta = 1_{GD}$. So these squares will commute iff we have commuting diagrams of natural transformations:

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta\star G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\circ\varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, Rightarrow, "F\circ\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon\star F"] \\ & F \end{tikzcd}$$

Here $\eta \star G$ denotes $(\eta_{GD}|D \in \mathcal{D}_0)$ and $G \circ \varepsilon$ denotes $(G(\varepsilon_D)|D \in \mathcal{D}_0)$. We refer to these as the *triangle equalities*.

We conclude:

Proposition 9.3 *Given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $\eta: 1_{\mathcal{C}} \Rightarrow GF$ *and* $\varepsilon: FG \Rightarrow 1_{\mathcal{D}}$ *satisfying* $(G \circ \varepsilon) \cdot (\eta \star G) = 1_G$ *and* $(\varepsilon \star F) \cdot (F \circ \eta) = 1_F$*, we have an adjunction* $F \dashv G$ *with* η *as its unit and* ε *as its counit.*

Proof. Because η and ε are natural transformations, the mappings

$$m_{C,D}(\alpha) := G\alpha \circ \eta_C \quad \text{ and } m^{-1}_{C,D}(\beta) = \varepsilon_D \circ F\beta$$

are natural; and because the triangle equalities hold, these operations are indeed each others inverses. ∎

For this reason an adjunction is also often defined as a pair of natural transformations $\eta: 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon: FG \Rightarrow 1_{\mathcal{D}}$ satisfying the triangle equalities.

Proposition 9.4 *Suppose that* $F: \mathcal{C} \to \mathcal{D}$ *is an equivalence of categories with pseudo-inverse* $G: \mathcal{D} \to \mathcal{C}$ *and natural isomorphisms* $\mu: 1_{\mathcal{C}} \Rightarrow GF$ *and* $\nu: FG \Rightarrow 1_{\mathcal{D}}$*. Then we can find an adjunction* $F \dashv G$ *whose unit is* μ*, and an adjunction whose counit is* ν *(but not necessarily both at the same time).*

Proof. By duality it suffices to prove the statement for μ. But if μ is a natural isomorphism and G is full and faithful, then

$$m_{C,D}(\alpha) := G\alpha \circ \mu_C$$

is a natural isomorphism. ∎

Note that, again by duality, we have that if F is an equivalence with pseudo-inverse G, then $G \dashv F$ as well.

9.3 Preservation of (co)limits by adjoint functors

A very important, and useful, aspect of adjoint functors is their behaviour with respect to limits and colimits.

Theorem 9.5 *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *such that* $F \dashv G$*. Then:*

(a) F *preserves all colimits which exist in* $\mathcal{C}$*;*

(b) G *preserves all limits which exist in* $\mathcal{D}$*.*

By duality it suffices to prove (a), which will follow from the following two lemmas.

Lemma 9.6 *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *such that* $F \dashv G$*. If* $\mathcal{C}$ *has an initial object, then it is preserved by* F*.*

Proof. Suppose I is initial in $\mathcal{C}$. To prove that FI is initial in $\mathcal{D}$, it suffices to prove that $\mathrm{Hom}_{\mathcal{D}}(FI, D)$ is a one-element set for any object D in $\mathcal{D}$. But this set is in bijective correspondence with $\mathrm{Hom}_{\mathcal{C}}(I, GD)$, which contains precisely one element, because I is initial in $\mathcal{C}$. ∎

Lemma 9.7 *Given* $F: \mathcal{C} \to \mathcal{D}$*, and* $M: \mathcal{E} \to \mathcal{C}$*. If* F *has a right adjoint, then the functor*

$$\widehat{F}: \mathit{Cocone}(M) \to \mathit{Cocone}(FM)$$

induced by F *has a right adjoint as well.*

Proof. Let $G: \mathcal{D} \to \mathcal{C}$ be right adjoint to F. The right adjoint to $\widehat{F}$ is the functor $\widetilde{G}$, which sends a cocone $(D, (\tau_E: FME \to D)_{E \in \mathcal{E}_0}))$ to $(GD, (\overline{\tau_E}: ME \to GD)_{E \in \mathcal{E}_0}))$, and whose action on a cocone morphisms is the same as applying G. Using the equation $Gg \circ \overline{\alpha} = \overline{g \circ \alpha}$ one quickly checks that this is indeed a functor.

To see that $\widetilde{G}$ is right adjoint to $\widehat{F}$ it suffices to check that for any pair of cocones $(C, (\sigma: FE \to C)_{E \in \mathcal{E}_0})$ and $(D, (\tau_E: FME \to D)_{E \in \mathcal{E}_0}))$, a map $\alpha: FC \to D$ is a map of cocones $\widehat{F}(C, \sigma) \to (D, \tau)$ if and only if $\overline{\alpha}: C \to GD$ is a map of cocones $(C, \sigma) \to \widetilde{G}(D, \tau)$. But the former statement says $\tau_E = \alpha \circ F\sigma_E$ and the latter $\overline{\tau_E} = \overline{\alpha} \circ \sigma_E$; so this follows from the equation $\overline{\alpha \circ Ff} = \overline{\alpha} \circ f$. ∎

From the theorem on preservation of (co)limits by adjoint functors one can often conclude that certain functors cannot have a right or a left adjoint.

Examples

a) The forgetful functor Mon $\to$ Set does not preserve epis, as we have seen in 1.2. In chapter 3 we've seen that f is epi iff

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "f"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "1"] & \bullet \end{tikzcd}$$

is a pushout; since left adjoints preserve identities and pushouts, they preserve epis; therefore the forgetful functor Mon $\to$ Set does not have a right adjoint;

b) The functor $(-) \times X\colon \mathrm{Set} \to \mathrm{Set}$ does not preserve the terminal object unless X is itself terminal in Set; therefore, it does not have a left adjoint for non-terminal X.

Another use of the theorem has to do with the computation of limits. Many categories, as we have seen, have a forgetful functor to Set which has a left adjoint. So the forgetful functor preserves limits, and since these can easily be computed in Set, one already knows the "underlying set" of the vertex of the limiting cone one wants to compute.

9.4 Exercises

Exercise 64 Let $\mathcal{C}$ and $\mathcal{D}$ be categories and

$$\Delta\colon \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$$

be the diagonal functor (see Chapter 3). Show that Δ has a left adjoint if and only if $\mathcal{D}$ has colimits of shape $\mathcal{C}$; and a right adjoint if and only if $\mathcal{D}$ has limits of shape $\mathcal{C}$.

Exercise 65 Given $\mathcal{C} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathcal{E}$, if $F_1 \dashv G_1$ and $F_2 \dashv G_2$ then $F_2F_1 \dashv G_1G_2$.

Exercise 66 Suppose that $F\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ is a functor, such that for $F^{\mathrm{op}}\colon \mathrm{Set} \to \mathrm{Set}^{\mathrm{op}}$ we have that $F^{\mathrm{op}} \dashv F$. Then there is a set A such that F is naturally isomorphic to $\mathrm{Set}(-, A)$.

Exercise 67 Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and that for each object D of $\mathcal{D}$ there is an object GD of $\mathcal{C}$ and an arrow $\varepsilon_D\colon FGD \to D$ with the property that for every object C of $\mathcal{C}$ and any map $f\colon FC \to D$, there is a unique $\tilde{f}\colon C \to GD$ such that

$$\begin{tikzcd} FC \arrow[rr, "f"] \arrow[dr, "F\tilde{f}"'] & & D \\ & FGD \arrow[ur, "\varepsilon_D"'] & \end{tikzcd}$$

commutes.

Prove that $G\colon \mathcal{D}_0 \to \mathcal{C}_0$ extends to a functor $G\colon \mathcal{D} \to \mathcal{C}$ which is right adjoint to F, and that $(\varepsilon_D\colon FGD \to D | D \in \mathcal{D}_0)$ is the counit of the adjunction.

Construct also the unit of the adjunction.

Exercise 68 Given $G\colon \mathcal{D} \to \mathcal{C}$, for each object C of $\mathcal{C}$ we let $(C{\downarrow}G)$ denote the category which has as objects pairs (D, g) where D is an object in $\mathcal{D}$ and $g\colon C \to GD$ is an arrow in $\mathcal{C}$. An arrow $(D, g) \to (D', g')$ in $(D{\downarrow}G)$ is an arrow $f\colon D \to D'$ in $\mathcal{D}$ which makes

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ GD \arrow[rr, "Gf"'] & & GD' \end{tikzcd}$$

commute.

Show that G has a left adjoint if and only if for each D, the category $(D{\downarrow}G)$ has an initial object.

Exercise 69 (Uniqueness of adjoints) Any two left (or right) adjoints of a given functor are naturally isomorphic. (*Hint:* Yoneda!)

Exercise 70 Suppose $\mathcal{D}$ has both an initial and a terminal object; denote by L the functor $\mathcal{D} \to \mathcal{D}$ which sends everything to the initial, and by R the one which sends everything to the terminal object. Prove that $L \dashv R$.

Exercise 71 An object M of a category $\mathcal{C}$ is called *injective* if for any diagram

$$\begin{tikzcd} A \arrow[d, "m"'] \arrow[dr, "f"] & \\ B & M \end{tikzcd}$$

with m a monomorphism, there exists an arrow $g\colon B \to M$ satisfying $gm = f$.

a) Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ functors with $F \dashv G$. Prove: if F preserves monos, then G preserves injective objects.

b) Formulate the statement dual to part a) (the dual notion of 'injective' is *projective*).

c) Now assume that in $\mathcal{D}$, for any object X there is an injective object M and a monomorphism $m\colon X \to M$ (one says: the category $\mathcal{D}$ *has enough injectives*). Prove the converse of part a).

Exercise 72 Show that the forgetful functor Pos $\to$ Set has a left adjoint, but not a right adjoint.

Hint: think of the coequalizer of the following two maps $\mathbb{Q} \to \mathbb{R}$ in Pos: one is the inclusion, the other is the constant zero map.

10 Monads and Algebras

10.1 Monads and adjunctions

Given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ let us look at the functor $T = GF: \mathcal{C} \to \mathcal{C}$.

We have seen that there is a natural transformation $\eta: 1_{\mathcal{C}} \Rightarrow T$. In addition, there is a natural transformation $\mu: T^2 \Rightarrow T$ whose components μ_C are

$$T^2(C) = GFGFC \overset{G(\varepsilon_{FC})}{\to} GFC = T(C).$$

Lemma 10.1 *The equalities*

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

hold. Here $(T\mu)_C = T(\mu_C): T^3C \to T^2C$ *and* $(\mu T)_C = \mu_{TC}: T^3C \to T^2C$ *(similarly for* ηT *and* $T\eta$*).*

Proof. We have

$$\mu_C \circ \eta_{TC} = G(\varepsilon_{FC}) \circ \eta_{GFC} = G(\varepsilon_{FC}) \circ \overline{1_{FC}} = \overline{\varepsilon_{FC}} = 1$$

and

$$\mu_C \circ T\eta_C = G(\varepsilon_{FC}) \circ GF\eta_C = G(\varepsilon_{FC} \circ F\eta_C) = 1,$$

where we have used the triangle equality in the second calculation. In addition, we have

$$\begin{aligned} \mu_C \circ T\mu_C = G(\varepsilon_{FC}) \circ GFG(\varepsilon_{FC}) = G(\varepsilon_{FC} \circ FG\varepsilon_C) =^{(*)} \\ G(\varepsilon_{FC} \circ \varepsilon_{FGFC}) = G\varepsilon_{FC} \circ G(\varepsilon_{FGCG}) = \mu_C \circ \mu_{TC}, \end{aligned}$$

where at (*) we have used the naturality of ε. ■

Definition 10.2 Let $\mathcal{C}$ be a category. A triple (T, η, μ) consisting of an endofunctor $T: \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta: 1_{\mathcal{C}} \Rightarrow T$ and $\mu: T^2 \Rightarrow T$ satisfying the identities in the previous lemma is called a *monad*.

Try to see the formal analogy between the defining equalities for a monad and the axioms for a monoid: writing $m(e, f)$ for ef in a monoid, and η for the unit element, we have

$$\begin{array}{ll} m(e, m(g, h)) = m(m(e, g), h) & \text{(associativity)} \\ m(\eta, e) = m(e, \eta) = e & \text{(unit)} \end{array}$$

Following this one calls μ the *multiplication* of the monad, and η its *unit*.

Example. The powerset functor $\mathcal{P}\colon \mathrm{Set} \to \mathrm{Set}$ (example h) of 2.2, with η and μ indicated there) is a monad (check).

Dually, there is the notion of a *comonad* (L, δ, ε) on a category $\mathcal{D}$, with equalities

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"'] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "1_L"'] \arrow[d, "\delta"] \arrow[dr, "1_L"] & \\ L & L^2 \arrow[l, "\varepsilon L"] \arrow[r, "L\varepsilon"'] & L \end{tikzcd}$$

Given an adjunction $(F, G, \varepsilon, \eta)$, $(FG, \delta = F\eta G, \varepsilon)$ is a comonad on $\mathcal{D}$. We call δ the *comultiplication* and ε the *counit* (this is in harmony with the unit-counit terminology for adjunctions).

Although, in many contexts, comonads and the notions derived from them are at least as important as monads, the treatment is dual so I concentrate on monads.

Every adjunction gives rise to a monad; conversely, every monad arises from an adjunction, but in more than one way. Essentially, there are a “maximal” (more precisely: terminal) and a “minimal” (more precisely: initial) solution to the problem of finding an adjunction from which a given monad arises.

Example. A monad on a poset P is a monotone function $T\colon P \to P$ with the properties $x \leq T(x)$ and $T^2(x) \leq T(x)$ for all $x \in P$; such an operation is also often called a *closure operation* on P. Note that $T^2 = T$ because T is monotone.

In this situation, let $Q \subseteq P$ be the image of T, with the ordering inherited from P. We have maps $r\colon P \to Q$ and $i\colon Q \to P$ such that ri is the identity on Q and $ir = T\colon P \to P$.

For $x \in P$, $y \in Q$ we have $x \leq i(y) \Leftrightarrow r(x) \leq y$ (check); so $r \dashv i$ and the operation T arises from this adjunction.

10.2 Algebras for a monad

Definition 10.3 Given a monad (T, η, μ) on a category $\mathcal{C}$, we define the category T-Alg of *(Eilenberg-Moore) algebras for the monad T* or *T-algebras* as follows:

- *Objects* are pairs (X,h) where X is an object of $\mathcal{C}$ and $h\colon T(X)\to X$ is an arrow in $\mathcal{C}$ such that

$$\begin{tikzcd} T^2(X) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\ T(X) \arrow[r, "h"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & T(X) \arrow[d, "h"] \\ & X \end{tikzcd}$$

  commute;

- *Morphisms*: $(X,h)\to(Y,k)$ are morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ for which

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "h"'] & T(Y) \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  commutes.

Theorem 10.4 *There is an adjunction between T-Alg and $\mathcal{C}$ which brings about the given monad T.*

Proof. There is an obvious forgetful functor $U^T\colon T\text{-Alg}\to\mathcal{C}$ which takes (X,h) to X. I claim that U^T has a left adjoint F^T:

F^T assigns to an object X the T-algebra $T^2(X) \xrightarrow{\mu_X} T(X)$; to $X \xrightarrow{f} Y$ the map $T(f)$; this is an algebra map because of the naturality of μ. That $T^2(X) \xrightarrow{\mu_X} T(X)$ is an algebra follows from the defining axioms for a monad T.

We have a natural transformation $\eta\colon 1\Rightarrow U^TF^T$, and to prove that it is the unit of an adjunction it suffices to prove that for each map $g\colon X\to U^T(Y,h)$, with X being an object X in $\mathcal{C}$ and (Y,h) being a T-algebra, there is a unique map of algebras $\overline{g}\colon F^TX\to(U,h)$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & U^TF^TX \arrow[d, "U^T(\overline{g})"] \\ & U^T(U,h) \end{tikzcd}$$

commutes. We put $\overline{g}\colon(T(X),\mu_X)\to(Y,h)$ to be the arrow $T(X) \xrightarrow{T(g)} T(Y) \xrightarrow{h} Y$. This is a map of algebras since

$$\begin{tikzcd} T^2(X) \arrow[r, "T^2(g)"] \arrow[d, "\mu_X"'] & T^2(Y) \arrow[r, "T(h)"] \arrow[d, "\mu_Y"'] & T(Y) \arrow[d, "h"] \\ T(X) \arrow[r, "T(g)"'] & T(Y) \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes; the left hand square is the naturality of μ; the right hand square is because (Y, h) is a T-algebra. In addition, we have

$$h \circ Tg \circ \eta_X = h \circ \eta_T \circ g = g,$$

since η is natural and h is an algebra.

To see that $\overline{g}$ is unique with this property, let $f: (TX, \mu_X) \to (Y, h)$ be a map of algebras with $U^T f \circ \eta_X = g$. Then

$$\overline{g} = h \circ Tg = h \circ Tf \circ T\eta_X =^{(*)} f \circ \mu_X \circ T\eta_X = f,$$

where at (*) we have used that f is a morphism of algebras. This finishes the proof of the existence of the adjunction with unit η.

The counit of the adjunction is the natural transformation which at (Y, h) is the transpose of $1_{U^T(Y,h)}$: that is, it is h considered as a map of algebras $(TY, \mu_Y) \to (Y, h)$. Therefore

$$T^2 = U^T F^T U^T F^T \stackrel{U^T \varepsilon F^T}{\to} U^T F^T = T$$

is the natural transformation which at X is the algebra structure on $F^T X$, which is μ_X. Hence the monad induced by the adjunction $F^T \dashv U^T$ is the monad T we started from. ∎

Example. The *group monad*. Combining the forgetful functor $U: \mathrm{Grp} \to \mathrm{Set}$ with the left adjoint, the free functor $\mathrm{Set} \to \mathrm{Grp}$, we get the following monad on Set:

$T(A)$ is the set of sequences on the alphabet $A \sqcup A^{-1}$ (A^{-1} is the set $\{a^{-1} | a \in A\}$ of formal inverses of elements of A, as in example e) of 1.1) in which no aa^{-1} or $a^{-1}a$ occur. The unit $A \stackrel{\eta_A}{\to} TA$ sends $a \in A$ to the string $\langle a \rangle$. The multiplication $\mu: T^2(A) \to T(A)$ works as follows. Define $(-)^-: A \sqcup A^{-1} \to A \sqcup A^{-1}$ by $a^- = a^{-1}$ and $(a^{-1})^- = a$. Define also $(-)^-$ on strings by $(a_1 \dots a_n)^- = a_n^- \dots a_1^-$. Now for an element of $TT(A)$, which is a string on the alphabet $T(A) \sqcup T(A)^{-1}$, say $\sigma_1 \dots \sigma_n$, we let $\mu_A(\sigma_1 \dots \sigma_n)$ be the concatenation of the strings $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$ on the alphabet $A \sqcup A^{-1}$, where $\tilde{\sigma}_i = \sigma_i$ if $\sigma_i \in T(A)$, and $\tilde{\sigma}_i = (\sigma_i)^-$ if $\sigma_i \in T(A)^{-1}$. Of course we still have to remove possible substrings of the form aa^{-1} etc.

Now let us look at algebras for the group monad: maps $T(A) \stackrel{h}{\to} A$ such that for a string of strings

$$\alpha = \sigma_1, \dots, \sigma_n = \langle \langle s_1^1, \dots, s_1^{k_1} \rangle, \dots, \langle s_n^1, \dots, s_n^{k_n} \rangle \rangle$$

we have that

$$h(\langle h\sigma_1, \dots, h\sigma_n \rangle) = h(\langle s_1^1, \dots, s_1^{k_1}, \dots, s_n^1, \dots, s_n^{k_n} \rangle)$$

and

$$h(\langle a \rangle) = a \text{ for } a \in A$$

I claim that this is the same thing as a group structure on A, with multiplication $a \cdot b = h(\langle a, b \rangle)$.

The unit element is given by $h(\langle\rangle)$; the inverse of $a \in A$ is $h(\langle a^{-1} \rangle)$ since

$$\begin{array}{rcl} h(\langle a, h(\langle a^{-1} \rangle)\rangle) & = & h(\langle h(\langle a \rangle), h(\langle a^{-1} \rangle)\rangle) = \\ h(\langle a, a^{-1} \rangle) & = & h(\langle\rangle), \text{ the unit element} \end{array}$$

Try to see for yourself how the associativity of the monad and its algebras transforms into associativity of the group law.

This situation is very important and has its own name:

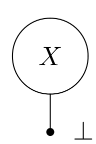
Definition 10.5 Given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ with $F \dashv G$, there is always a *comparison functor* $K: \mathcal{D} \to T\text{-Alg}$ for $T = GF$, the monad induced by the adjunction. K sends an object D of $\mathcal{D}$ to the T-algebra $GFG(D) \stackrel{G(\varepsilon_D)}{\to} G(D)$, and a map $f: D \to D'$ to Gf. (One can use one of the triangle equalities and naturality of ε to prove this is well-defined.)

We say that the functor $G: \mathcal{D} \to \mathcal{C}$ is *monadic*, or by abuse of language (if G is understood), that $\mathcal{D}$ *is monadic over* $\mathcal{C}$, if K is an equivalence.

In many cases however, the situation is not monadic. Take the forgetful functor $U: \text{Pos} \to \text{Set}$. It has a left adjoint F which sends a set X to the discrete ordering on X ($x \leq y$ iff $x = y$). Of course, UF is the identity on Set and the UF-algebras are just sets. The comparison functor K is the functor U, and this is not an equivalence.

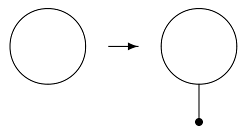
Another example of a monadic situation is of importance in domain theory. Let $\text{Pos}_\perp$ be the category of partially ordered sets with a least element, and order preserving maps which also preserve the least element.

There is an obvious inclusion functor $U: \text{Pos}_\perp \to \text{Pos}$, and U has a left adjoint F. Given a poset X, $F(X)$ is X "with a bottom element added":

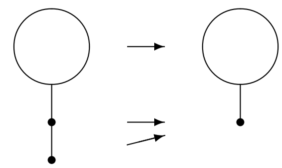


Given $f: X \to Y$ in Pos, $F(f)$ sends the new bottom element of X to the new bottom element of Y, and is just f on the rest. If $f: X \to Y$ in Pos is a map and Y has a least element, we have $F(X) \to Y$ in $\text{Pos}_\perp$ by sending $\perp$ to the least element of Y. So the adjunction is clear.

The monad $UF: \text{Pos} \to \text{Pos}$, just adding a least element, is called the *lifting monad*. Unit and multiplication are:



$\eta_X : X \to T(X)$



$\mu_X : T^2(X) \to T(X)$

A T-algebra $h: TX \to X$ is first of all a monotone map, but since $h\eta_X = 1_X$, h is epi in Pos so surjective. It follows that X must have a least element $h(\perp)$. From the axioms for an algebra one deduces that h must be the identity when restricted to X, and $h(\perp)$ the least element of X.

Another poset example: algebras for the power set monad $\mathcal{P}$ on Set (example j) of 2.2). Such an algebra $h: \mathcal{P}(X) \to X$ must satisfy $h(\{x\}) = x$ and for $\alpha \subseteq \mathcal{P}(X)$:

$$h(\{h(A) | A \in \alpha\}) = h(\{x | \exists A \in \alpha (x \in A)\}) = h(\bigcup \alpha)$$

Now we can, given an algebra structure on X, define a partial order on X by putting $x \leq y$ iff $h(\{x, y\}) = y$.

Indeed, this is clearly reflexive and antisymmetric. As to transitivity, if $x \leq y$ and $y \leq z$ then

$$\begin{array}{rl} h(\{x,z\}) = & h(\{x, h(\{y,z\})\}) \quad = \\ h(\{h(\{x\}), h(\{y,z\})\}) = & h(\{x\} \cup \{y,z\}) \quad = \\ h(\{x,y\} \cup \{z\}) = & h(\{h(\{x,y\}), h(\{z\})\}) = \\ h(\{y,z\}) = & z \end{array}$$

so $x \leq z$.

Furthermore this partial order is *complete*: least upper bounds for arbitrary subsets exist. For $\bigvee B = h(B)$ for $B \subseteq X$: for $x \in B$ we have $h(\{x, h(B)\}) = h(\{x\} \cup B) = h(B)$ so $x \leq \bigvee B$; and if $x \leq y$ for all $x \in B$ then

$$\begin{array}{rl} h(\{h(B), y\}) = & h(B \cup \{y\}) \quad = \\ h(\bigcup_{x \in B} \{x, y\}) = & h(\{h(\{x,y\}) | x \in B\}) = \\ h(\{y\}) = & y \end{array}$$

so $\bigvee B \leq y$.

We can also check that a map of algebras is a $\bigvee$-preserving monotone function. Conversely, every $\bigvee$-preserving monotone function between complete posets determines a $\mathcal{P}$-algebra homomorphism.

We have an equivalence between the category of complete posets and $\bigvee$-preserving functions, and $\mathcal{P}$-algebras.

10.3 T-algebras at least as complete as $\mathcal{C}$

Let T be a monad on $\mathcal{C}$. The following exercise is meant to show that if $\mathcal{C}$ has all limits of a certain type, so does T-Alg. In particular, if $\mathcal{C}$ is complete, so is T-Alg; this is often an important application of a monadic situation. In fact, something stronger holds.

Definition 10.6 For a functor $G: \mathcal{D} \to \mathcal{C}$ we say that G *creates limits* of type $\mathcal{E}$ if for every functor $M: \mathcal{E} \to \mathcal{D}$ and every limiting cone (C, σ) for GM in $\mathcal{C}$, there is a unique cone (D, τ) for M in $\mathcal{D}$ which is taken by G to (C, σ), and moreover this unique cone is limiting for M in $\mathcal{D}$.

Clearly, if G creates limits of type $\mathcal{E}$ and $\mathcal{C}$ has all limits of type $\mathcal{E}$, then $\mathcal{D}$ has them, too.

Proposition 10.7 *The forgetful functor $U^T: T - \mathrm{Alg} \to \mathcal{C}$ creates limits of every type.*

Proof. Let $M: \mathcal{E} \to T$-Alg be functor and (C, σ) be a limiting cone for $U^T M$. For objects E of $\mathcal{E}$, let $M(E)$ be the T-algebra $T(m_E) \stackrel{h_E}{\to} m_E$, so that $\sigma_E: C \to m_E$.

We need to construct a map $h: TC \to C$ turning all σ_E into maps of algebras, meaning that $\sigma_E \circ h = h_E \circ T\sigma_E$. Since (C, σ) is a limiting cone on $U^T M$, this means that we have to prove that $(TC, (h_E \circ T\sigma_E)_{E \in \mathcal{E}})$ is a cone on $U^T M$. But $f: E \to E'$ is a map in $\mathcal{E}$ and $g = U^T M f: m_E \to m_{E'}$, then

$$g \circ h_E \circ T\sigma_E = h_{E'} \circ Tg \circ T\sigma_E = h_{E'} \circ T\sigma_{E'}.$$

So there is a unique map $h: TC \to C$ such that $\sigma_E \circ h = h_E \circ T\sigma_E$.

We need to show that h gives C a T-algebra structure, so $h \circ \eta_Y = 1_C$ and $h \circ \mu = h \circ Th$. To prove these equalities, it suffices to prove that the left and right hand side become equal after precomposing with all maps of the form σ_E (C being a limit). Therefore

$$\sigma_E \circ h \circ \eta_Y = h_E \circ T\sigma_E \circ \eta_Y = h_E \circ \eta_{m_E} \circ \sigma_E = \sigma_E,$$

shows the first equality. The proof of the second equality is similar.

It remains to verify that h is indeed the limit in T-Alg. We leave this to the reader. ∎

10.4 Reflective subcategories

In this subsection we will take a closer look at adjunctions where the right adjoint is full and faithful. As it turns out this is equivalent to the counit being a natural isomorphism.

Proposition 10.8 *Let $G: \mathcal{D} \to \mathcal{C}$ be a functor with left adjoint $F: \mathcal{C} \to \mathcal{D}$ and counit $\varepsilon: FG \Rightarrow 1$. Then:*

(i) G is faithful if and only if every component of ε is epi.

(ii) G is full if and only if every component of ε is split mono.

(iii) G is full and faithful if and only if ε is a natural isomorphism.

Proof. Let us first observe that for any map $f: A \to B$ in $\mathcal{D}$ the map

$$FGA \xrightarrow{\varepsilon_A} A \xrightarrow{f} B$$

transposes under the adjunction to Gf:

$$\overline{f \circ \varepsilon_A} = Gf \circ \overline{\varepsilon_A} = Gf \circ 1_A = Gf.$$

(i) Note that for any pair of parallel maps $f, g: A \to B$ in $\mathcal{D}$, we have:

$$Gf = Gg \Leftrightarrow \overline{Gf} = \overline{Gg} \Leftrightarrow f \circ \varepsilon_A = g \circ \varepsilon_A.$$

G being faithful means that the statement on the left implies $f = g$ for any pair of parallel maps f, g in $\mathcal{D}$, while ε being pointwise epi means that the statement on the right implies $f = g$ for any pair of parallel maps f, g in $\mathcal{D}$. This shows (i).

(ii, $\Rightarrow$) Let A be any object in $\mathcal{D}$, and consider $\eta_{GA}: GA \to GFGA$. If G is full, there is a map $\alpha: A \to FGA$ with $G\alpha = \eta_{GA}$. Then

$$\overline{\alpha \circ \varepsilon_A} = G\alpha = \eta_{GA} = \overline{1_{FGA}},$$

and hence $\alpha \circ \varepsilon_A = 1_{FGA}$. So ε is pointwise split mono.

(ii, $\Leftarrow$) Let A and B be objects in $\mathcal{D}$, $g: GA \to GB$ be any map in $\mathcal{C}$ and $\alpha: A \to FGA$ be such that $\alpha \circ \varepsilon_A = 1_{FGA}$: the aim is to find a map $f: A \to B$ such that $Gf = g$. So let us put $f := \varepsilon_B \circ Fg \circ \alpha$. Since $\alpha \circ \varepsilon_A = 1_{FGA}$, we have

$$G\alpha = \overline{\alpha \circ \varepsilon_A} = \overline{1_{FGA}} = \eta_{GA},$$

and therefore, using the triangle equalities,

$$Gf = G\varepsilon_B \circ GFg \circ G\alpha = G\varepsilon_B \circ GFg \circ \eta_{GA} = G\varepsilon_B \circ \eta_{GB} \circ g = g,$$

as desired.

(iii) follows from (i) and (ii). ∎

Corollary 10.9 *Let $G: \mathcal{D} \to \mathcal{C}$ be a full and faithful right adjoint. If $\mathcal{C}$ has colimits of shape $\mathcal{E}$, then so does $\mathcal{D}$.*

Proof. Write F for the left adjoint to G and note that the previous result tells us that $FG \cong 1$. Let $M: \mathcal{E} \to \mathcal{D}$ be a diagram and C be the colimit for GM in $\mathcal{C}$. Since F preserves colimits, FC will be the colimit for $FGM \cong M$. ∎

Proposition 10.10 *If $G: \mathcal{D} \to \mathcal{C}$ is a full and faithful right adjoint, then G is monadic.*

Proof. Let F be be the left adjoint, T be the induced monad and

$$K: \mathcal{D} \to T\text{-Alg}$$

be the comparison functor. Since G is full and faithful, K is full and faithful as well; so it remains to show that K is essentially surjective.

Let us first observe that for any pair of objects A, B in $\mathcal{C}$ we have a commutative diagram of sets:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FA, FB) \arrow[rr, "f \mapsto \overline{f}"] \arrow[dr, "f \mapsto Gf"'] & & \mathrm{Hom}_{\mathcal{C}}(A, GFB) \\
& \mathrm{Hom}_{\mathcal{C}}(GFA, GFB) \arrow[ur, "g \mapsto g \circ \eta_A"'] &
\end{tikzcd}$$

Since both the top and left arrow are bijections, so must be the arrow on the right.

Next, we will show that $h: TX \to X$ will be T-algebra if and only η_X is an iso and h is its inverse. If $h: GFX \to X$ is a T-algebra, then $h \circ \eta_X = 1$. But then $\eta_X \circ h \circ \eta_X = \eta_X$, so the previous observation gives $\eta_X \circ h = 1$ as well. Conversely, η_X is invertible and h is its inverse, then $h \circ \eta_X = 1$. Moreover, $\mu_X \circ T\eta_X = 1$ implies $h \circ \mu_X = h \circ Th$.

So if $h: TX \to X$ is a T-algebra, h is an iso and the commutativity of

$$\begin{tikzcd}
GFGFX \arrow[r, "GFh"] \arrow[d, "\mu = G(\varepsilon_{FX})"] & GFX \arrow[d, "h"] \\
GFX \arrow[r, "h"] & X
\end{tikzcd}$$

shows that K is an equivalence and FU^T is its pseudo-inverse. ∎

Definition 10.11 A category $\mathcal{D}$ is a *subcategory* of the category $\mathcal{C}$ if there is a functor $G: \mathcal{D} \to \mathcal{C}$ such that the underlying maps $\mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ar}(\mathcal{D}) \to \mathrm{Ar}(\mathcal{C})$ are inclusions. A subcategory $\mathcal{D}$ is called *replete* if for any isomorphism $f: C \to D$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ we have that both C and f belong to $\mathcal{D}$. A subcategory $\mathcal{D}$ is *full* if G is full (it is, of course, always faithful). A subcategory is called *reflective* if it is full, replete and G has a left adjoint (this left adjoint is then called the *reflector*).

Proposition 10.12 *Let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ with inclusion $G: \mathcal{D} \to \mathcal{C}$. Then G creates limits.*

Proof. Let us write F for the reflector and T for the monad induced on $\mathcal{C}$. In this case the comparison functor

$$K: \mathcal{D} \to T\text{−Alg}$$

is an isomorphism. We already now that it is full and faithful, so it remains to show that it induces a bijection on objects. Since G is injective on objects, the same applies to K; so it remains to show that K is surjective on objects. But if $h: GFX \to X$ is a T-algebra, h will be isomorphism, as we have seen. Since $GFX \in \mathcal{D}$, we have $X \in \mathcal{D}$ by repleteness. So the statement follows Proposition 10.7. ∎

10.5 Exercises

Exercise 73 Finish the proof of the theorem: for the group monad T, there is an equivalence of categories between T-Alg and Grp. Check that the functor T-Alg $\to$ Grp defined there is a pseudo inverse to the comparison functor K.

Exercise 74 Let $P: \text{Set}^{\text{op}} \to \text{Set}$ be the contravariant powerset functor, and $\bar{P}$ its left adjoint, as in j) of 5.1. Let $T: \text{Set} \to \text{Set}$ the induced monad.

a) Describe unit and multiplication of this monad explicitly.

b) Show that Set^{op} is equivalent to T-Alg [Hint: if this proves hard, have a look at VI.4.3 of Johnstone's "Stone Spaces"].

c) Conclude that there is an adjunction

$$\text{CABool} \rightleftarrows \text{Set}$$

which presents CABool as monadic over Set.

Exercise 75 Let Rng1 be the category of rings with unit and unitary ring homomorphisms. Since every ring with 1 is a (multiplicative) monoid, there is a forgetful functor $G: \text{Rng1} \to \text{Mon}$. For a monoid M, let $Z[M]$ be the ring of formal expressions

$$n_1c_1 + \cdots + n_kc_k$$

with $k \geq 0$, $n_1, \ldots, n_k \in Z$ and $c_1, \ldots, c_k \in M$. This is like a ring of polynomials, but multiplication uses the multiplication in M. Show that this defines a functor $F: \text{Mon} \to \text{Rng1}$ which is left adjoint to G, and that G is monadic, i.e. the category of GF-algebras is equivalent to Rng1. [Hint: Proceed as in the example of the powerset monad. That is, let $h: GF(M) \to M$ be a monoid homomorphism which gives M the structure of a GF-algebra. Find an abelian group structure on M such that M becomes a ring with unit]

Given a monad T on a category $\mathcal{C}$, let us define a category T-Adj of adjunctions $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ such that $GF = T$. A *map* of such T-adjunctions from $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ to $\mathcal{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{D}'$ is a functor $K: \mathcal{D} \to \mathcal{D}'$ satisfying $KF = F'$ and $G'K = G$.

Exercise 76 For the purposes of this exercise let us write $\mathcal{C}^T$ for the category T-Alg and write $G^T: T\text{-Alg} \to \mathcal{C}$ for the forgetful functor. In case T arises from an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, there was a comparison functor $\mathcal{D} \xrightarrow{K} \mathcal{C}^T$.

(a) Show that in the diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "K"] \arrow[dr, shift left, "G"'] & & \mathcal{C}^T \arrow[dl, shift left, "G^T"] \\
& \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"'] &
\end{tikzcd}$$

we have that $KF = F^T$ and $G^T K = G$.

(b) Show that the functor K is unique with this property. In other words, $\mathcal{C}^T$ together with F^T and G^T is the terminal object in T-Adj.

Exercise 77 In this exercise we will construct the initial object in T-Adj: the *Kleisli category* of T, called $\mathcal{C}_T$. $\mathcal{C}_T$ has the same objects as $\mathcal{C}$, but a map in $\mathcal{C}_T$ from X to Y is an arrow $X \xrightarrow{f} T(Y)$ in $\mathcal{C}$. Composition is defined as follows: given $X \xrightarrow{f} T(Y)$ and $Y \xrightarrow{g} T(Z)$ in $\mathcal{C}$, considered as a composable pair of morphisms in $\mathcal{C}_T$, the composition gf in $\mathcal{C}_T$ is the composite

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T^2(Z) \xrightarrow{\mu_Z} T(Z)$$

in $\mathcal{C}$.

(a) Prove that $\mathcal{C}_T$ defined this way is category.

(b) The adjunction $\mathcal{C}_T \underset{G_T}{\overset{F_T}{\leftrightarrows}} \mathcal{C}$ is defined as follows: the functor G_T sends the object X to $T(X)$ and $f: X \to Y$ ($f: X \to T(Y)$ in $\mathcal{C}$) to

$$T(X) \xrightarrow{T(f)} T^2(Y) \xrightarrow{\mu_Y} T(Y)$$

The functor F_T is the identity on objects and sends $X \xrightarrow{f} Y$ to $X \xrightarrow{f} Y \xrightarrow{\eta_Y} T(Y)$, considered as $X \to Y$ in $\mathcal{C}_T$. Show that F_T and G_T are functors and check $F_T \dashv G_T$ by constructing the unit and counit.

(c) Now let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction $F \dashv G$ with $GF = T$. Show that there is a unique comparison functor $L: \mathcal{C}_T \to \mathcal{D}$ such that $GL = G_T$ and $LF_T = F$.

Hint: L sends the object X to $F(X)$ and $f: X \to Y$ (so $f: X \to T(Y) = GF(Y)$ in $\mathcal{C}$) to its transpose $\tilde{f}: F(X) \to F(Y)$.

Exercise 78 What does the Kleisli category for the covariant powerset monad look like?

Exercise 79 Let T be a monad on $\mathcal{C}$. Call an object of T-Alg *free* if it is in the image of $F^T: \mathcal{C} \to T - \text{Alg}$. Show that the Kleisli category $\mathcal{C}_T$ is equivalent to the full subcategory of T-Alg on the free T-algebras.

11 Presheaves revisited

In this final section we return to the theory of presheaves.

11.1 The category of elements

The purpose of this section is to show that every presheaf is a colimit of representables.

Let $\mathbb{C}$ be a small category and X be a presheaf over $\mathbb{C}$. Recall that we have a category $y \downarrow X$:

Objects An object consists of a pair (C, x) where C is an object in the category $\mathbb{C}$ and x is a morphism $x: yC \to X$.

Morphisms A morphism $(D, y) \to (C, x)$ is a morphism $\alpha: D \to C$ such that $x \circ y\alpha = y$.

This category is called the *category of elements of* X (and sometimes also denoted by $\int_{\mathbb{C}} X$). Note that it is a small category because $\mathbb{C}$ is small.

Remark 11.1 Note that under the Yoneda Lemma, a morphism $x: yC \to X$ corresponds to an element $x \in X(C)$. So we can also think of the objects of the category of elements as pairs (C, x) such that $x \in X(C)$. In that case a morphism $(D, y) \to (C, x)$ is a morphism $\alpha: D \to C$ such that $x \cdot \alpha = y$. In what follows we will routinely identify maps $yC \to X$ and elements $x \in X(C)$.

There is an obvious forgetful functor $U: y \downarrow X \to \mathbb{C}$ which we can compose with the Yoneda embedding to obtain a functor

$$y \circ U: y \downarrow X \to \mathrm{Psh}(\mathbb{C}).$$

We can think of this functor as a diagram. By construction there is a cocone on this diagram with vertex X: indeed, for each object (C, x) in $y \downarrow X$ there is a morphism $yU(C, x) = yC \to X$ which is x. Let us write π for this cocone $y \downarrow X \Rightarrow \Delta(X)$.

Proposition 11.2 *This cocone is colimiting. Therefore every presheaf is a colimit of representables.*

Proof. Suppose Y is a presheaf and we have a cocone $\rho: y \circ U \Rightarrow \Delta(Y)$. This cocone ρ chooses for each $x \in X(C)$ a map $\rho_{(C,x)}: yC \to Y$ which we can think of as an element of $Y(C)$. So for our morphism of cocones $F: X \to Y$ we put:

$$F_C(x) = \rho_{(C,x)}.$$

Since we want $F \circ \pi_{(C,x)} = \sigma_{(C,x)}$ this definition is forced. Indeed, once we show that F is a natural transformation, this equation will follow from the Yoneda Lemma.

So let $x \in X(C)$ and $\alpha: D \to C$ be given. Then:

$$\begin{array}{rcll} F_D(x \cdot \alpha) & = & \rho_{(D, x\cdot\alpha)} & \\ & = & \rho_{(C,x)} \circ y\alpha & (\rho \text{ cocone}) \\ & = & F_C(x) \cdot \alpha. & \end{array}$$

This shows that $F: X \to Y$ is a natural transformation, which also completes the proof. ∎

11.2 Kan extensions

Theorem 11.3 *For any small category $\mathbb{C}$, the Yoneda embedding*

$$y: \mathbb{C} \to \mathrm{PSh}(\mathbb{C})$$

has the following universal property: given a category $\mathcal{E}$ which is locally small and cocomplete and a functor $F: \mathbb{C} \to \mathcal{E}$, there is a colimit preserving functor $F_!: \mathrm{PSh}(\mathbb{C}) \to \mathcal{E}$ such that $F_! \circ y \cong F$ as indicated in the following diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "y"] \arrow[dr, "F"'] & \mathrm{PSh}(\mathbb{C}) \arrow[d, dotted, "F_!"] \\ & \mathcal{E} \end{tikzcd}$$

Moreover, up to natural isomorphism, $F_!$ is the unique colimit preserving functor with this property.

We refer to $F_!$ as the *left Kan extension* of F.

Proof. If $F_!$ preserves colimits and $F_! \circ y \cong F$, then the following definition is forced:

$$F_!(X) = \mathrm{colim}\,(F \circ U: y \downarrow X \to \mathbb{C} \to \mathcal{E}).$$

In the sequel we will also just write:

$$F_!(X) = \mathrm{colim}_{(C,x) \in y \downarrow X} FC.$$

In particular, for each (C, x) we have a map $\varphi^X_{(C,x)}: FC \to F_!X$. In addition, if $K: X \to Y$ is a map of presheaves, we have a cocone on $F \circ U: y \downarrow X \to \mathcal{E}$ whose component at (C, x) is $\varphi^Y_{(C, K_C(x))}: FC \to F_!C$. Therefore there exists a unique map $F_!(K): F_!X \to F_!Y$ such that

$$F_!K \circ \varphi^X_{(C,x)} = \varphi^Y_{(C,K_C(x))}.$$

From this it follows that $F_!$ is a functor. We clearly have that $F_! \circ y \cong F$, since $y \downarrow yC$ has a terminal object (which is $(C, 1_C)$) and the colimit of a diagram with a terminal object is the value at that terminal object. In addition, this shows that $\varphi^X_{(C,x)} \cong F_!(x\colon yC \to X)$, which also tells us that our definition of $F_!K$ was forced.

To show that $F_!$ preserves colimits, we show something stronger: we show that $F_!$ has a right adjoint $F^*\colon \mathcal{E} \to \mathrm{PSh}(\mathbb{C})$. Indeed, the Yoneda Lemma allows us to guess the formula for F^*. Since we must have

$$F^*(E)(C) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(yC, F^*E) \cong \mathrm{Hom}_{\mathcal{E}}(F_!(yC), E) \cong \mathrm{Hom}_{\mathcal{E}}(FC, E),$$

we put

$$F^*(E)(C) := \mathrm{Hom}_{\mathcal{E}}(FC, E)$$

and we have

$$\mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(yC, F^*E) \cong \mathrm{Hom}_{\mathcal{E}}(FC, E)$$

by construction.

More formally, we have a functor

$$\mathbb{C}^{\mathrm{op}} \times \mathcal{E} \xrightarrow{F^{\mathrm{op}} \times 1} \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \xrightarrow{\mathrm{Hom}} \mathcal{S}ets$$

which transposes to a functor $F^*\colon \mathcal{E} \to \mathrm{PSh}(\mathbb{C})$.

Now for any presheaf X we have a bijective correspondence:

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(X, F^*E) & = & \mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(\mathrm{colim}_{(C,x)\in y\downarrow X}\, yC, F^*E) \\
& \cong & \lim_{(C,x)\in y\downarrow X} \mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(yC, F^*E) \\
& \cong & \lim_{(C,x)\in y\downarrow X} \mathrm{Hom}_{\mathcal{E}}(FC, E) \\
& \cong & \mathrm{Hom}_{\mathcal{E}}(\mathrm{colim}_{(C,x)\in y\downarrow X}\, FC, E) \\
& = & \mathrm{Hom}_{\mathcal{E}}(F_!X, E).
\end{array}$$

We leave the proof that this correspondence is natural to the reader. ■

Example 11.4 Simplicial sets are an important structure in algebraic topology: they can be defined as presheaves on a category called Δ. This category Δ looks as follows:

Objects: The objects of this category are sets of the form $[n] = \{0, 1, \ldots, n\}$ where n is a natural number. We think of these as finite non-empty linear orders (carrying the usual ordering as natural numbers).

Morphisms: Maps $[m] \to [n]$ are monotone functions.

There is a functor $\Delta \to \mathrm{Top}$ sending $[n]$ to the n-simplex:

$$\Delta^n = \{\, (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0, \text{ for all } i \,\}.$$

(Note that the 0-simplex is just a point; the 1 -simplex is a line segment; the 2-simplex is triangle; the 3-simplex is tetrahedron, et cetera.) If $f\colon [m] \to [n]$ is a monotone function, then this induces a continuous map $\Delta(f)\colon \Delta^m \to \Delta^n$ as follows:

$$\Delta(f)(x_0, \ldots, x_m)_i = \sum \{x_j : 0 \leq j \leq m, f(j) = i\}.$$

The previous proposition says that these data determine an adjunction between the category $\widehat{\Delta}$ of simplicial sets and the category Top of topological spaces. The left adjoint $\widehat{\Delta} \to \mathrm{Top}$ is called *geometric realisation*, while the right adjoint $\mathrm{Top} \to \widehat{\Delta}$ takes the *singular (simplicial) complex* of a topological space.

Theorem 11.5 *Let $f\colon \mathbb{C} \to \mathbb{D}$ be a functor between small categories. The precomposition functor*

$$f^*\colon \widehat{\mathbb{D}} \to \widehat{\mathbb{C}}$$

given by $f^*(Q)(C) = Q(fC)$ has both adjoints.

Proof. Writing $F = y \circ f\colon C \to \widehat{\mathbb{D}}$, the proof of the previous theorem gives one an adjunction $F_! \dashv F^*$ with

$$F^*(Q)(C) = \mathrm{Hom}_{\widehat{\mathbb{D}}}(FC, Q).$$

But since

$$\mathrm{Hom}_{\widehat{\mathbb{D}}}(FC, Q) = \mathrm{Hom}_{\widehat{\mathbb{D}}}(yfC, Q) \cong Q(fC)$$

by the Yoneda Lemma, we have $F^* \cong f^*$ and we deduce that f^* has a left adjoint. So it remains to show that f^* has a right adjoint f_*.

The formule for f_* can again be deduced from the Yoneda Lemma. Indeed we must have:

$$f_*(X)(D) \cong \mathrm{Hom}_{\widehat{\mathbb{D}}}(yD, f_*X) \cong \mathrm{Hom}_{\widehat{\mathbb{C}}}(f^*yD, X),$$

so we put $f_*(X)(D) := \mathrm{Hom}_{\widehat{\mathbb{C}}}(f^*yD, X)$. Note that $f_*(yD)(C) = \mathrm{Hom}_{\mathbb{D}}(fC, D)$, so we can construct f_* more formally by first taking the transpose $J\colon \mathbb{D} \to \widehat{\mathbb{C}}$ of

$$\mathrm{Hom} \circ (f^{\mathrm{op}} \times 1)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{S}ets$$

and then defining $f_*\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ to be transpose of

$$\mathrm{Hom} \circ (J^{\mathrm{op}} \times 1)\colon \mathbb{D}^{\mathrm{op}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}^{\mathrm{op}} \times \widehat{\mathbb{C}} \to \mathcal{S}ets.$$

To prove that $f^* \dashv f_*$, we use that f^* preserves colimits because these are computed pointwise in presheaf categories. Therefore we have a bijective correspondence:

$$\begin{aligned}
\mathrm{Hom}_{\widehat{\mathbb{C}}}(f^*Q, X) &\cong \mathrm{Hom}_{\widehat{\mathbb{D}}}(f^*(\mathrm{colim}_{(D,q)\in y\downarrow Q}\, yD), X) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{C}}}(\mathrm{colim}_{(D,q)\in y\downarrow Q}\, f^*yD, X) \\
&\cong \lim_{(D,q)\in y\downarrow Q} \mathrm{Hom}_{\widehat{\mathbb{C}}}(f^*yD, X) \\
&\cong \lim_{(D,q)\in y\downarrow Q} \mathrm{Hom}_{\widehat{\mathbb{D}}}(yD, f_*X) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{D}}}(\mathrm{colim}_{(D,q)\in y\downarrow Q}\, yD, f_*X) \\
&\cong \mathrm{Hom}_{\widehat{\mathbb{D}}}(Q, f_*X).
\end{aligned}$$

The proof that this correspondence is natural is left to the reader. (Lecturer needs a break.) ∎

11.3 Exercises

Exercise 80 Let $\mathbb{C}$ be a small category and X be a presheaf over $\mathbb{C}$. Show that

$$\mathrm{PSh}(\mathbb{C})/X \cong \mathrm{PSh}(y \downarrow X).$$

In other words: presheaf categories are closed under slicing.